FIRST-ORDER COMBINATORICS AND A DEFINITION TO THE CONCEPT OF A MODEL-THEORETIC PROPERTY WITH DEMONSTRATION OF POSSIBLE APPLICATIONS

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The work is devoted to the first-order combinatorics presenting a conceptual foundation for investigations concerned the expressive power of predicate logic. We give a definition to the concept of a model-theoretic property, and specify in detail the pragmatic approach that turns out to be the most adequate to the real practice of investigations in model theory.

Keywords: first-order logic, Tarski-Lindenbaum algebra, Hanf's isomorphism of theories, model-theoretic property.

In the work [1], initial notions of the first-order combinatorics are defined presenting a conceptual framework for investigations of expressive power of predicate logic. First-order combinatorics requires to accept some family of methods of transformation of theories. A transformation has the aim either to simplify the theory or to reduce it to a definite form with the same isomorphism type of the Tarski-Lindenbaum algebra and with preserving model-theoretic properties of corresponding completions. Although the common practice of investigations in model theory widely uses a terminology connected with the model-theoretic properties of complete theories, however, this term was not specified in any way yet. Thus, a necessity to give a definition to the concept of a model-theoretic property becomes actual.

Combinatorics of a given type is characterized by a definite set of used methods of transformation of theories and by the layer of those model-theoretic properties which are preserved by these methods. Relation between the accepted class of methods and the semantic layer of preserved model-theoretic properties is a Galois's correspondance. Therefore, an inverse dependence takes place between the set of methods and the volume of the layer of preserved on them model-theoretic properties. Signature reduction procedures and transformations by the universal construction of finitely axiomatizable theories are considered as combinatorial methods in predicate logic; these methods are taken as a basis for the first-order combinatorics. For finitary combinatorics, we accept methods of transformation of finite signatures and more general methods of Cartesian extensions of theories, while for infinitary combinatorics, we accept methods of reduction of infinite signatures to finite ones as well as transformations of theories by means of the universal construction. The class of finitary methods can separately be considered. As for the class of infinitary methods, its consideration requires adding the finitary methods.

In this work, we describe the operation of a Cartesian extension of a theory, give a definition to the concept of a model-theoretic property, and specify in detail the pragmatic approach that turns out to be the most adequate to the real practice of investigations in model theory. Although the definition of a model-theoretic property includes some informal parts, nevertheless, its applications ensure exact mathematical statements.

Preliminaries. We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, constructive models, and Boolean algebras that can be found in [2], [3], and [4]. A signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n \geqslant 2$, or two unary function symbols. In the work, the signatures are considered only, which admit Gödel's numbering of the formulas. Such a signature is called *enumerable*. Generally, *incomplete theories* are considered. For theories, c.a means *computably axiomatizable*, while f.a. means *finitely axiomatizable.*

There are two levels of definability in first-order logic. The first one is called *radically logical* or briefly *model*. It does not assume any limitation on the class of used formulas. The second more delicate level is called *algebraic*. At this level, $\exists\cap\forall$-type of first-order definability is used. In this work, we systematically follow the algebraic approach. If it is needed, all results in the article can be transferred to the form corresponding the model-type definability.

1 Cartesian-type interpretations

We use a simplest concept of an *interpretation* of a theory T_0 in the region $U(x)$ of a theory T_1, [5]. Classes of *isostone* and *model-bijestive* interpretations are introduced in [6]. In this section, we introduce a technical class of interpretations presenting finitary methods in first-order logic.

Given a signature σ and a finite sequence of formulas of this signature of either of the following forms:

$$\begin{aligned} &\texttt{(a)}\ \varkappa = \langle \varphi_1^{m_1}/\varepsilon_1, \varphi_2^{m_2}/\varepsilon_2, \ldots, \varphi_s^{m_s}/\varepsilon_s \rangle, \\ &\texttt{(b)}\ \varkappa = \langle \varphi_1^{m_1}, \varphi_2^{m_2}, \ldots, \varphi_s^{m_s} \rangle, \end{aligned} \tag{1.1}$$

where φ_k is a formula with m_k free variables, $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ is a formula with $2m_k$ free variables such that $\mathrm{Len}\,\bar{y}_k = \mathrm{Len}\,\bar{z}_k = m_k$; moreover, (1.1)(b) is just a simpler notation instead of the common entry (1.1)(a) in the case when $\varepsilon_k(\bar{y}_k, \bar{z}_k)$ coincides with $\bar{y}_k = \bar{z}_k$ for all $k \leqslant s$.

Starting from a model $\mathfrak{M}$ of signature σ together with a tuple $\varkappa$ of any of the forms (1.1)(a,b), we are going to construct a new model $\mathfrak{M}_1 = \mathfrak{M}\langle\varkappa\rangle$ of signature

$$\sigma_1 = \sigma \cup \quad \{U^1, U_1^1, U_2^1, \ldots, U_s^1\} \cup \{K_1^{m_1+1}, \ldots, K_s^{m_s+1}\} \tag{1.2}$$

as follows. As the universe, we take $|\mathfrak{M}_1| = |\mathfrak{M}| \cup A_1 \cup A_2 \cup \ldots \cup A_s$, where all specified parts are pairwise disjoint sets. On the set $|\mathfrak{M}|$, all symbols of signature σ are defined exactly as they were defined in $\mathfrak{M}$; in the remainder, they are defined trivially; predicate $U(x)$ distinguishes $|\mathfrak{M}|$; predicate $U_k(x)$ distinguishes A_k; the other predicates are defined by specific rules depending on the case. In the case (1.1)(b), each predicate K_k in (1.2) should be defined so that it would represent a one-to-one correspondence between the set of tuples $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}$ and the set $A_k = U_k(\mathfrak{M}_1)$. Turn to the most common case (1.1)(a). Denote by $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ a sentence stating that ε_k is an equivalence relation on the set of tuples distinguished by the formula $\varphi_k(\bar{x})$ in $\mathfrak{M}$. In this case, (m_k+1)-ary predicate K_k should be defined so that it would represent a one-to-one correspondence between the quotient set $\{\bar{a} \mid \mathfrak{M} \models \varphi_k(\bar{a})\}/\varepsilon'_k$ and the set $U_k(\mathfrak{M}_1)$, where $\varepsilon'_k(\bar{y}, \bar{z}) = \varepsilon_k(\bar{y}, \bar{z}) \vee \neg\mathrm{Equiv}(\varepsilon_k, \varphi_k)$. The aim of replacement of ε_k by ε'_k using $\mathrm{Equiv}(\varepsilon_k, \varphi_k)$ is to provide total definiteness of the operation of an extension $\mathfrak{M}\langle\varkappa\rangle$ independently of whether the formulas ε_k represent equivalence relations in corresponding domains or not. In the case (1.1)(a), $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian-quotient extension of* $\mathfrak{M}$, while in the case (1.1)(b), the model $\mathfrak{M}\langle\varkappa\rangle$ is said to be a *Cartesian extension of* $\mathfrak{M}$ *by a sequence of formulas* $\varkappa$.

Expand the operation of an extension (initially defined for models) on theories. Given a theory T and a tuple $\varkappa$ of the form (1.1). Using a fixed signature (1.2) for extensions of models, we define a new theory $T' = T\langle\varkappa\rangle$ as follows: $T' = \mathrm{Th}(K)$, $K = \{\mathfrak{M}\langle\varkappa\rangle \mid \mathfrak{M} \in \mathrm{Mod}(T)\}$. In the case (1.1)(a) it is called a *Cartesian-quotient extension*, while in the case (1.1)(b) it is called a *Cartesian extension* of T *by a sequence* $\varkappa$.

Normally, we follow an algebraic approach; i.e., we consider passages $T \mapsto T\langle\varkappa\rangle$ for which the sequence (1.1) satisfies the following technical condition:

$$\varphi_k(\bar{x}_k) \text{ and } \varepsilon_k(\bar{y}_k, \bar{z}_k) \text{ are } \exists\cap\forall\text{-presentable, for all } k \leqslant s. \tag{1.3}$$

Denote by $\mathcal{K}D(\sigma)$ and $\mathcal{K}C(\sigma)$ the sets of tuples of formulas of signature σ of the forms, respectively, (1.1)(a) and (1.1)(b), while $\mathcal{K}D$ and $\mathcal{K}C$ are unions of these sets for all possible (enumerable) signatures σ. We denote by $\mathcal{K}C_{\exists\cap\forall}$ the set of all tuples (1.1)(b) satisfying (1.3), while $\mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$ is the set of all tuples (1.1)(a) satisfying (1.3).

In theory $T\langle\varkappa\rangle$, the region $U(x)$ represents a model of theory T. Particularly, the transformation $T \mapsto T\langle\varkappa\rangle$ defines a natural interpretation $I_{T,\varkappa}$ of T in $T\langle\varkappa\rangle$. It is called a *special Cartesian-quotient* interpretation. Similar definition applies to the other case of the tuple $\varkappa$; thereby, the concepts of a *special Cartesian* interpretation is also defined. Considering theories up to an algebraic isomorphism, we may use simpler term *Cartesian-quotient* or, respectively, *Cartesian* interpretation.

Lemma 1.1. *Given a theory T of an enumerable signature σ and a sequence of formulas $\varkappa \in \mathcal{K}D(\sigma)$. Special Cartesian-quotient interpretation $I_{T,\varkappa} : T \rightarrowtail T\langle\varkappa\rangle$ is effective, model-bijective, and isostone. In particular, interpretation $I_{T,\varkappa}$ determines a computable isomorphism $\mu_{T,\varkappa} : \mathcal{L}(T) \to \mathcal{L}(T\langle\varkappa\rangle)$ between the Tarski-Lindenbaum algebras.*

The following statement is established based on first-order combinatorial properties of Cartesian extensions of theories:

Lemma 1.2. *The following relation defined on the class of all theories*

$$T \widetilde{\approx}_a S \Leftrightarrow_{dfn} (\exists \varkappa' \varkappa'' \in \mathcal{K}C_{\exists\cap\forall}) \left[T\langle\varkappa'\rangle \approx_a S\langle\varkappa''\rangle \right] \tag{1.4}$$

is reflexive, symmetric, and transitive (*i.e., it is an equivalence relation*).

Further properties of Cartesian-type extensions of theories and interpretations can be found in [7] and [8].

Definition 1.A. We introduce the following notations for particular semantic layers that are relevant in this direction:

(A) ASL = the set of model-theoretic properties $\mathfrak{p} \in AL$ preserved by any special Cartesian interpretation $I_{T,\xi} : T \rightarrowtail T\langle\xi\rangle$ for an *arbitrary computably* *axiomatizable theory T of an enumerable signature* σ and an arbitrary finite tuple $\xi=\langle\varphi_1,...,\varphi_s\rangle$ of sentences of signature σ satisfying (1.3).

(B) $MSL = ASL \cap ML$.

(C) $ACL =$ the set of model-theoretic properties $\mathfrak{p} \in AL$ preserved by any special Cartesian interpretation $I_{T,\xi}: T \rightarrowtail T\langle\xi\rangle$ for an *arbitrary computably axiomatizable theory T of an enumerable signature* σ and an arbitrary tuple $\xi=\langle\varphi_1^{m_1},\dots,\varphi_s^{m_s}\rangle$ of formulas of signature σ satisfying (1.3).

(D) $MCL = ACL \cap ML$.

(E) $ADL =$ the set of model-theoretic properties $\mathfrak{p} \in AL$ preserved by any special Cartesian-quotient interpretation $I_{T,\xi}: T \rightarrowtail T\langle\xi\rangle$ for an *arbitrary computably axiomatizable theory T of an enumerable signature* σ and an arbitrary tuple $\xi=\langle\varphi_1^{m_1}/\varepsilon_1,\dots,\varphi_s^{m_s}/\varepsilon_s\rangle$ of formulas of signature σ satisfying (1.3).

(F) $MDL = ADL \cap ML$.

Layer *ACL* is said to be the (*algebraic*) *Cartesian* semantic layer; it plays the role of a *pragmatic release* of the *finitary semantic layer*. By *MCL* we denote its model version called the *model Cartesian* layer. Layer *ADL* is said to be the (*algebraic*) *Cartesian-quotient* semantic layer; it plays the role of a *maximalistic release* of the *finitary semantic layer*. By *MDL*, we denote its model version called the *model-type Cartesian-quotient* layer.

Fig. 1 presents a scheme of inclusions between the semantic layers and corresponding similarity relations relevant for first-order combinatorics. Arrows point out relatively stronger similarity relations and relatively wider semantic layers of model-theoretical properties. Two relations $\approx$ and $\approx_a$ in the top are relations of isomorphism of theories, where $\approx$ means a *model isomorphism* or simply *isomorphism*, while $\approx_a$ means an *algebraic isomorphism* or $\exists\cap\forall$-*presentable equivalence* between theories. Although $\approx$ and $\approx_a$ are not similarity relations, they are included in the scheme for the sake of completeness. The entries $\equiv_c$, $\equiv_{ac}$, etc., are short forms for semantic similarity relations $\equiv_{MCL}$, $\equiv_{ACL}$ with semantic layers *MCL*, *ACL*, etc., that were defined above. The inclusions $MDL \subseteq MCL$ and $ADL \subseteq ACL$ are also valid although they are not presented in the scheme in Fig. 1.

The layer *MQL* consists of the model-theoretic properties preserved by all interpretations in the class $IQuasi \cup ICartes$ between computably axiomatizable theories, where *IQuasi* is the set of all quasiexact interpretations, while *ICartes* is the set of all Cartesian interpretations. The layer *MQL* is supported by a regular version of the universal construction of finitely axiomatizable theories, [6].

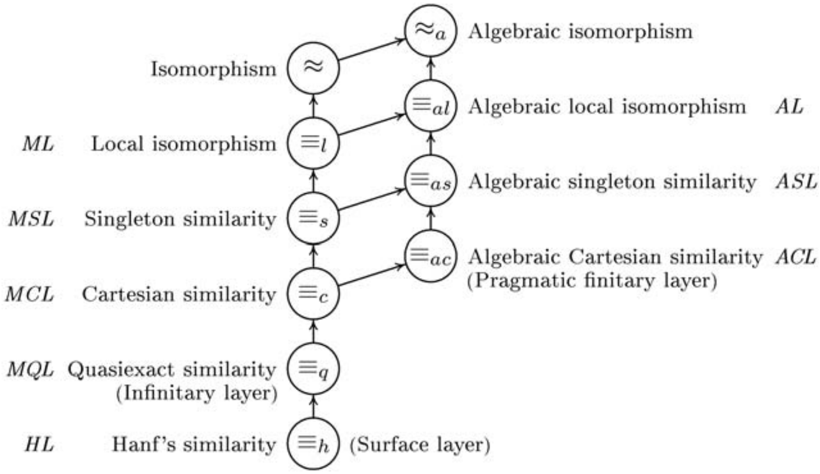

Figure 1 — Scheme of semantic layers of model-theoretic properties

The Hanf semantic layer *HL* is an empty set $\varnothing$. Corresponding semantic similarity relation $\equiv_\varnothing$, alternatively $\equiv_h$, is called *Hanf's isomorphism* because William Hanf was the first investigator who studied such relations between theories just in relation to the problem of expressive possibilities of first-order logic, [9].

2 A definition to the concept of a model-theoretic property

We are going to discuss approaches to the problem of classification of complete theories modulo coincidence of their model-theoretic properties, cf. [10]. Two complete theories are said to be *equivalent* if their real model-theoretic properties are identical:

$$T_1 \overset{\mathrm{MT}}{\simeq} T_2 \Leftrightarrow_{dfn} (\forall \textit{ real model-theoretic property } \mathfrak{p})\big[T_1 \in \mathfrak{p} \Leftrightarrow T_2 \in \mathfrak{p}\big]. \qquad (2.1)$$

Accordingly, any classes of complete theories closed under $\overset{\mathrm{MT}}{\simeq}$ are said to be *real* model-theoretic properties. Thus, to define the concept of a real model-theoretic property it is necessary to find available dependencies (called reasoning) between complete theories of the following form

$$T_1 \simeq_{\mathrm{x}} T_2 \Rightarrow T_1 \overset{\mathrm{MT}}{\simeq} T_2, \qquad (2.2)$$

that have significance in the practice of working in model theory.

Two most important reasoning (for complete theories) are:

$$\texttt{(a)}\ T \approx_a S \Rightarrow T \overset{\mathrm{MT}}{\simeq} S, \qquad (2.3)$$

$$\texttt{(b)}\ T\langle\varkappa\rangle = S \Rightarrow T \overset{\mathrm{MT}}{\simeq} S, \textit{ for any } \varkappa \in \mathcal{K}C_{\exists\cap\forall}.$$

General significance of the reasoning (2.3)(a) is obvious. Argumentation (2.3)(b) concerns virtual expansions of the universe which are just plain codings for the initial universe; therefore, the pointed out sequence of implications (2.3)(b) for all $\varkappa \in \mathcal{K}C$ can also be considered as adequate to the common practice of work in model theory. Notice that, lots of researchers follow a naive approach considering any classes of complete theories, even if they are not closed under isomorphisms of theories. To avoid this common irregular situation, we will assume (by default) that any considered class of complete theories first should be closed under algebraic isomorphisms of theories by the rule

$$\mathfrak{p} \;\mapsto\; \mathfrak{p}^* = [\mathfrak{p}]_{\approx_a} = \{T \in \mathbb{C} \mid (\exists T' \in \mathfrak{p})\,[T \approx_a T']\}. \tag{2.4}$$

This correction rule is said to be a *normalization pre-stage* in the definition we are going to introduce.

We give a *generic definition* to the concept of a model-theoretic property.

Definition 2.A. [Generic definition of a model-theoretic property].

Initially, we have to point out a collection of relations of reasoning of the form (for complete theories)

$$\simeq_{\mathrm{x}}^{(i)},\ i \in I \tag{2.5}$$

that we intend to accept as a basis of the definition. The relation $\overset{\mathrm{MT}}{\simeq}$, cf. (2.1), is presented by the relation $\simeq_{\mathrm{x}}^{*}$ obtained by the operation of closure of the system of relations (2.5) up to an equivalence relation. Accordingly, the class of all real model-theoretic properties is presented by the following expression:

$$AreaL = \{\mathfrak{p} \subseteq \mathbb{C} \mid \mathfrak{p} \textit{ is closed under } \simeq_{\mathrm{x}}^{*}\}. \tag{2.6}$$

To check up, whether a set $\mathfrak{p} \subseteq \mathbb{C}$ is a model-theoretic property, first, a normalization pre-stage $\mathfrak{p} \mapsto \mathfrak{p}^*$ should be performed; then, the condition $\mathfrak{p}^* \in AreaL$ is to be checked. If the result is positive, we qualify $\mathfrak{p}$ as a *real model-theoretic property*; moreover, a specifying term "$\mathfrak{p}$ is a model-theoretic property up to the closure under isomorphisms" may be used. Otherwise, if the test $\mathfrak{p}^* \in AreaL$ fails, $\mathfrak{p}$ is qualified as a class that is not a real model-theoretic property.

End of the definition.

Lemma 2.1. *An inverse dependence of the set of real model-theoretic properties on the accepted set of reasoning $\simeq_{\mathrm{x}}^{(i)}$, $i \in I$, takes place.*

Proof. Indeed, let the pointed out set defines an equivalence relation $\simeq_{\mathrm{x}}^{*}$ playing the role of the relation $\overset{\mathrm{MT}}{\simeq}$, thus, defining the layer $AreaL$. Assume that, as the base for a new definition, some larger set of reasoning $\simeq_{\mathrm{x}}^{(i)}$, $i \in I^{+}$,

$I^{+} \supseteq I$, is taken. It is obvious that the inclusion $\simeq^{*}_{\mathrm{x}} \subseteq \simeq^{+}_{\mathrm{x}}$ must take place; i.e., each class of the new equivalence $\simeq^{+}_{\mathrm{x}}$ consists of a number of classes of the initial equivalence $\simeq^{*}_{\mathrm{x}}$. Thereby, we have $AreaL^{+} \subseteq AreaL$ because $AreaL^{+}$ consists of the sets of complete theories closed under equivalence $\simeq^{+}_{\mathrm{x}}$ having larger classes in comparison with those of the initial equivalence $\simeq^{*}_{\mathrm{x}}$. □

The following (pragmatic) variant of the definition is fixed as preferable:

Definition 2.B. [Pragmatic specification to Definition 2.A].

As a set of reasoning, we accept the relation (2.3)(a) together with a series of relations (2.3)(b) for all $\varkappa \in \mathcal{K}C_{\exists\cap\forall}$. The relation $\approxeq_a$ on the class of all complete theories defined by expression (1.4) in Lemma 1.2 is the closure of this system of relations. Thus, within this approach, relation $\overset{\mathrm{MT}}{\simeq}$ coincides with $\approxeq_a$. Accordingly, in view of the scheme of semantic layers in Fig. 1, we obtain the following chain of inclusions:

$$AreaL = ACL \subseteq ASL \subseteq AL. \tag{2.7}$$

By default, we also suppose that, to apply Definition 2.B for a set $\mathfrak{p} \subseteq \mathbb{C}$, a normalization transformation (2.4) should be performed initially.

End of the definition.

An important statement concerning different versions of Definition 2.A.

Lemma 2.2. *Suppose that a variant α of definition of a real model-theoretic property is chosen with reasoning consisting of the relation* (2.3)(*a*) *and a series of relations* (2.3)(*b*) *for all $\varkappa \in \mathcal{K}C_{\exists\cap\forall}$ together with a definite set of additional relations of the form* (2.2). *Then, the following chain of inclusions takes place*:

$$AidealL^{\alpha} \subseteq AreaL^{\alpha} \subseteq ACL \subseteq ASL \subseteq AL, \tag{2.8}$$

where the ideal semantic layer $AidealL^{\alpha}$ corresponds to the potential possibility of an extension of the accepted system of reasoning α with some new rules of the form (2.2) *that can appear and could be accepted in the future within the system α.*

Proof. From the principle of inverse dependence we mentioned earlier. □

The following systems of reasoning to the definition of the concept of a real model-theoretic property are possible. Let an arbitrary set $\mathfrak{p} \subseteq \mathbb{C}$ be given. At the *naive approach*, any set of complete theories is considered as a model-theoretic property; the *primitive approach* requires that $\mathfrak{p}$ should be closed under isomorphisms of theories; the *pragmatic approach*, cf. Definition 2.B, requires that $\mathfrak{p}$ is closed under isomorphisms, Cartesian extensions, and back transitions in the operation of Cartesian extensions of theories; at last, the *maximalistic approach* requires that $\mathfrak{p}$ is closed under isomorphisms, Cartesian-quotient extensions, and back transitions in the operation of Cartesian-quotient extensions of theories, i.e., the reasoning $T\langle\varkappa\rangle = S \Rightarrow T \overset{\text{MT}}{\simeq} S$, for all $\varkappa \in \mathcal{K}D^{\varepsilon}_{\exists\cap\forall}$, is accepted that is wider in comparisons with (2.3)(b).

Generally speaking, other approaches for the definition to the concept of a realistic model-theoretic property are possible. They can be based on other principles different from those adopted within the proposed scheme. To compare the approach suggested here with the other potentially possible ones, some discussion may be needed about the advantages and disadvantages of each of the alternative approaches.

3 Transformations of theories complementary to methods of finitary and infinitary first-order combinatorics

In this section, we describe a number of interesting transformations of theories, which have properties close to those of methods of first-order combinatorics. These transformations are intended for different purposes, including, that they can be used as possible extensions of argumentation in a definition to the concept of a model-theoretic property. The suggested demonstrations represent a useful addition to the first-order combinatorial approach.

Demo 1. The *rounding operation* modulo theory *SI*. We consider transformation of theories $T \mapsto T \oplus SI$, where *SI* is a *successor* theory with an *initial element* and *without cycles*.

Signature $\sigma_{SI} = \{\lhd^2, c\}$, axioms of the theory *SI*:

1°. $(\forall x)\big[\neg(x \lhd x)\big]$,
2°. $(\forall x)(\forall y)(\forall z)\big[(x \lhd y) \,\&\, (x \lhd z) \to (y = z)\big]$,
3°. $(\forall x)(\forall y)(\forall z)\big[(y \lhd x) \,\&\, (z \lhd x) \to (y = z)\big]$,
4°. $(\forall x)(\exists y)\big[x \lhd y\big]$,
5°. $(\forall x)\big[(x \neq c) \to (\exists y)(y \lhd x)\big]$,
6°. $(\forall x)\big[\neg(x \lhd c)\big]$.
7°. $(\forall z_0 z_1 \ldots z_n)\big(z_0 \lhd z_1 \lhd \ldots \lhd z_n \to z_0 \neq z_n\big)$, $n \in \mathbb{N} \setminus \{0\}$.

The theory *SI* is so simple that, in the common practice, anyone can say that this theory has no model-theoretic properties. In view of its completeness and decidability, the passage $T \mapsto T \oplus SI$ preserves an isomorphism type of the Tarski-Lindenbaum algebra and transfers the majority of model theoretic properties to correspondent complete extensions of theories. The rounding operation, cf. Section 2.5 in [11], does not belong to the class of finitary methods as it does not preserve the property to have a finite model. It is possible to check that the nanural interpretation of $I\colon T\mapsto T\oplus SI$ is quasiexact. Therefore, this operation is an infinitary method by definition. In the work [6] the rounding operation is applied in the order to transform a theory into a theory with infinite models. This represents a useful reception allowing to prevent situations with finite models in constructions corresponding to the infinitary direction in model theory.

Demo 2. Continual series of rounding operations. Such series can be constructed based on the idea used in the proof of Lemma 1.4.1 in [11]. A set $A\subseteq\mathbb{N}$ is called rarefied if $A=\{n_0,n_1,\dots,n_i,\dots\}$, $n_0<n_1<\dots<n_i<\dots$; moreover, limit of the sequence $a_i=(n_{i+1}-n_i)$ is infinity. By $\mathfrak{A}$, we denote the family of all rarefied sets. It is obvious that $\mathfrak{A}$ is a continual family. Consider a fixed set $A\in\mathfrak{A}$. We construct a modification $SI[A]$ of theory SI of a successor relation as follows. Signature $\sigma_{SI[A]}=\{\lhd^2,c,U^1\}$, axioms of theory $SI[A]$ are the following:

1°. $(\forall x)\big[\neg(x\lhd x)\big]$,
2°. $(\forall x)(\forall y)(\forall z)\big[(x\lhd y)\,\&\,(x\lhd z)\to(y=z)\big]$,
3°. $(\forall x)(\forall y)(\forall z)\big[(y\lhd x)\,\&\,(z\lhd x)\to(y=z)\big]$,
4°. $(\forall x)(\exists y)\big[x\lhd y\big]$,
5°. $(\forall x)\big[(x\neq c)\to(\exists y)(y\lhd x)\big]$,
6°. $(\forall x)\big[\neg(x\lhd c)\big]$.
7°. $(\forall z_0z_1\dots z_n)\big(z_0\lhd z_1\lhd\dots\lhd z_n\to z_0\neq z_n\big)$, $n\in\mathbb{N}\setminus\{0\}$.
8°. $(\forall z_0z_1\dots z_n)\big(c=z_0\lhd z_1\lhd\dots\lhd z_n\to U(z_n)\big)$, for all $n\in A$.
9°. $(\forall z_0z_1\dots z_n)\big(c=z_0\lhd z_1\lhd\dots\lhd z_n\to\neg U(z_n)\big)$, for all $n\in\mathbb{N}\setminus A$.
10°. $(\forall z_0z_1\dots z_n)\big(U(z_0)\,\&\,z_0\lhd z_1\lhd\dots\lhd z_n\to\neg U(z_n)\big)$, for all $n\in\mathbb{N}\setminus\{0\}$.

Similarly to the case of rounding modulo SI, each transformation $T\mapsto T\oplus SI[A]$, $A\in\mathfrak{A}$, can be also considered as a rounding operation as it preserves an isomorphism type of the Tarski-Lindenbaum algebra and transfers the majority of model theoretic properties to correspondent completions of theories.

Demo 3. An alternative continual series of the rounding operations. By $\mathfrak{A}'$, we denote the family of sets $A\subseteq\mathbb{N}$ satisfying $\{0,1\}\cap A=\varnothing$. It is obvious that $\mathfrak{A}'$ is a continual family. Consider a fixed $A\in\mathfrak{A}$. We construct a modification $SI^\circ[A]$ of the successor theory SI° with cycles as follows. Signature $\sigma_{SI^\circ[A]}=\{\lhd^2,c,U^1\}$, axioms are the following sentences:

1°. $(\forall x)\big[\neg(x\lhd x)\big]$,
2°. $(\forall x)(\forall y)(\forall z)\big[(x\lhd y)\,\&\,(x\lhd z)\to(y=z)\big]$,
3°. $(\forall x)(\forall y)(\forall z)\big[(y\lhd x)\,\&\,(z\lhd x)\to(y=z)\big]$,

4°. $(\forall x)(\exists y)\big[x\lhd y\big]$,
5°. $(\forall x)\big[(x\neq c)\to(\exists y)(y\lhd x)\big]$,
6°. $(\forall x)\big[\neg(x\lhd c)\big]$.
7°. For any $n\in A$, there is the only $\lhd$-cycle of length n; moreover, exactly one element in the cycle satisfies $U(x)$.
8°. For any $n\in\mathbb{N}\setminus A$, there is no $\lhd$-cycles of length n.
9°. $(\forall z_1\ldots z_n)\big(c\lhd z_1\lhd\ldots\lhd z_n\to\neg U(z_n)\big)$, for all $n\in\mathbb{N}$.
10°. $(\forall z_0z_1\ldots z_n)\big(U(z_0)\,\&\,z_0\lhd z_1\lhd\ldots\lhd z_n\,\&\,U(z_n)\to z_0=z_n\big)$, for all $n\in\mathbb{N}\setminus\{0\}$.

Similarly to Demo 2, this series of transformation of theories is continual. Moreover, each of them can play the role of a rounding operation. This demo is based on the E.A.Palutin's remark having the aim to point out a collection consisting of the continuum of pairs of different theories having identical model-theoretic properties at author's talk at Maltsev's Meeting in November 2016 in Novosibirsk.

Demo 4. A quick scheme for the infinitary semantic layer. As it is noted in [12], the following series of transformations

$$\begin{array}{ll}\texttt{(a)} & T\mapsto T\langle\varkappa\rangle,\ \varkappa\in\mathcal{K}C,\\ \texttt{(b)} & T\mapsto T\langle\varkappa\rangle\oplus SI,\ \varkappa\in\mathcal{K}C,\end{array}\tag{3.1}$$

determines both finitary and infinitary semantic layers of model-theoretic properties. Namely, up to coincidence modulo a representative list $\mathcal{R}$ of model-theoretic properties, ACL is equal to the set of all $\mathfrak{p}\in ML$ such that $\mathfrak{p}$ is closed under $\approx_a$ and is preserved under any transformation (3.1)(a), while MQL is the set of all $\mathfrak{p}\in ML$ such that $\mathfrak{p}$ is closed under $\approx_a$ and is preserved under any transformation (3.1)(b) for an arbitrary sequence of formulas $\varkappa\in\mathcal{K}C$ and an arbitrary computably axiomatizable theory T. It seems, the practical rule (3.1)(b) is the simplest representation allowing to preliminary estimate either inclusion or non-inclusion of a property $\mathfrak{p}$ to the infinitary semantic layer MQL.

Demo 5. *Combinations of methods.* Different variants of the rounding operation preserve an infinitary layer, but do not that for finitary layer. On the other hand, two rules (3.1)(a) and (3.1)(b) correspond to finitary and infinitary types of first-order combinatorics. This suggests an idea to define a series of combined rules with different transformations as follows. Consider a fixed method $\mathfrak{m}'$ that is finitary or close to finitary, and a fixed method $\mathfrak{m}''$ that is infinitary or close to infinitary. We define their combination $\mathfrak{m}=\mathfrak{m}'\uplus\mathfrak{m}''$ on the class of complete theories T as follows

$$\mathfrak{m}(T)=\begin{cases}\mathfrak{m}'(T), & \textit{if } T \textit{ has a finite model},\\ \mathfrak{m}''(T), & \textit{if } T \textit{ is elementary theory of an infinite model}\end{cases}\tag{3.2}$$

Furthermore, the method $\mathfrak{m}$ that is initially defined on the class of all complete theories can be expanded to the class of all theories (including, incomplete ones) by the following rule:

$$
\begin{aligned}
T \simeq_a^{\mathfrak{m}} S \Leftrightarrow_{dfn} \quad & \big(\exists\ \textit{computable isomorphism}\ \mu : \mathcal{L}(T) \to \mathcal{L}(S)\big) \\
& (\forall\, \textit{complete extension}\ T' \supseteq T) \\
& (\forall\, \textit{complete extension}\ S' \supseteq S) \\
& [S' = \mu(T') \Rightarrow (S' \approx_a \mathfrak{m}(T'))].
\end{aligned}
\tag{3.3}
$$

These combined methods can be used as argumentation in the definition of a model-theoretic property.

Remark 3.1. Consider theory Sc that is a *successor* theory with a *distinguished element* and *without cycles* defined as follows. Signature $\sigma_{Sc} = \{\lhd^2, c\}$, axioms of the theory:

1°. $(\forall x)\big[\neg(x \lhd x)\big]$,

2°. $(\forall x)(\forall y)(\forall z)\big[(x \lhd y)\,\&\,(x \lhd z) \to (y = z)\big]$,

3°. $(\forall x)(\forall y)(\forall z)\big[(y \lhd x)\,\&\,(z \lhd x) \to (y = z)\big]$,

4°. $(\forall x)(\exists y)\big[x \lhd y\big]$,

5°. $(\forall y)(\exists x)\big[x \lhd y\big]$,

6°. $(\forall z_0 z_1 \ldots z_n)\big(z_0 \lhd z_1 \lhd \ldots \lhd z_n \to z_0 \neq z_n\big)$, $\ n \in \mathbb{N} \setminus \{0\}$.

It is possible to use theory Sc instead of SI in Demo 1, Demo 2, and other similar constructions, obtaining the same results. On the other hand, it is impossible to do without axioms preventing cycles in Demo 1, because an isomorphism of the Tarski-Lindenbaum algebras will not exist. Furthermore, it is impossible to do without constant c in Demo 1 based on theory S of signature $\{\lhd\}$ without cycles, because under an isomorphism of the Tarski-Lindenbaum algebras, the model-theoretic property of being a theory of a rigid model is not preserved.

4 Advantages of the pragmatic approach

A scheme of all possible variants of definition to the concept of a model-theoretic property is given in Fig. 2. The scheme presents a primitive approach (a), a restricted pragmatic approach (b) with the class of all methods used by the finite signature reduction procedure, a pragmatic approach (c) with the class of all Cartesian methods, a maximalistic approach (d) with the class of Cartesian-quotient methods, as well as different intermediate between (c) and (d) variants. Possible, some classes of finitary methods beyond the set of Cartesian-quotient extensions can be discovered in the future (the question of existence of such methods is open). Some examples in the preceding section can be accepted as additional argumentation to the definition, allowing us to create realistic systems with different variants of the concept of a model-theoretic property.

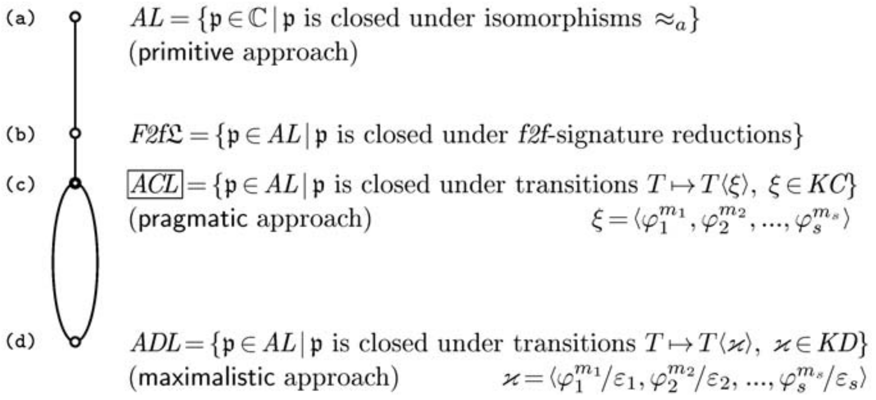

Fig. 2. Versions of the concept of a model-theoretic property

A general analysis of the definition of a model-theoretic property shows that the most significant is the variant (c) with a finitary layer *ACL* and with the set of methods, including all Cartesian extensions of theories. Indeed, the main results on expressive power of first-order logic are obtained based on the operation of a Cartesian extension of a theory, i.e., within the pragmatic approach (c). Extension of the argumentation with Cartesian-quotient extensions of theories is inexpedient as it leads to smaller semantic layer of model-theoretic properties in view of the principle of inverse dependence. It is also important that the operation of a Cartesian extension of a theory corresponds more to spirit of general model theory whereas the operation of a Cartesian-quotient extension has a certain algebraic accent. It shows the special significance of the pragmatic variant (c) of the concept of a model theoretic property.

The close variant (b) based on the class of finite signature reduction procedures preserves a semantic layer $F2f\mathcal{L}$ which does not differ from *ACL* modulo representative list $\mathcal{R}$, [1]. However, the variant (c) looks more fundamental in comparison with (b) since the former depends on arbitrariness while the choice of forms of coding configurations for the finite signature reduction procedure. As for the primitive approach (a), it defines a rather large layer of model-theoretic properties. However, there are no constructions supporting this layer so its role is strictly subordinated with a technical toolkit in investigations. As for the maximal approach (d), its incentive motive is to reach the bigger fundamental nature based on Cartesian-quotient extensions of theories representing the class of all finitary first-order methods, even despite certain reduction of corresponding semantic layer of model-theoretic properties.

Conclusion

In the work, we introduce a definition to the concept of a model-theoretic property. Subsequent demonstrations give a generalized view on the possible extensions of the concept of a model-theoretic property. Notice that, there is a possibility to create new argumentations by the method of generic extensions in set theory. At the same time, addition of any new argumentations to a pragmatic version of the definition of a model-theoretic property can only reduce the layer controlled by finitary first-order combinatorics. An important point is that although the suggested definition of the concept of a real model-theoretic property uses some informal parts, nevertheless, exact mathematical statements are obtained based on this definition.

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