Signature reduction procedures and the universal construction as transformation methods of theories within the first-order combinatorial approach

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Abstract

Работа развивает предложенный ранее комбинаторный подход первого порядка представляющий концептуальную основу для исследования структуры обобщённых алгебр Тарского-Линденбаума исчислений предикатов конечных богатых сигнатур над финитарным и инфинитарным семантическими слоями. В настоящей работе приведены ключевые технические спецификации определяющие статус процедур редукции сигнатур и универсальной конструкции конечно аксиоматизируемых теорий в рамках комбинаторного подхода первого порядка.

Ключевые слова: логика первого порядка, неполная теория, теоретико-модельное свойство, алгебра Тарского-Линденбаума, вычислимый изоморфизм, вычислимый класс теорий, универсальная конструкция конечно аксиоматизируемых теорий.

Results of the works [7] and [8] describe some special methods of constructing computable isomorphisms between the Tarski-Lindenbaum algebras of predicate calculi $PC(\sigma_1)$ and $PC(\sigma_2)$ of finite rich signatures σ_1 and σ_2. Finite-to-finite signature reduction procedure is involved in the main construction of the work [8]; it would be natural to call these transformations of theories as methods of *finitary first-order combinatorics*. On the other hand, an available version of the universal construction of finitely axiomatizable theories is involved in the proof of the main statement of the work [7]; thus, it would be natural to call these transformations of theories as methods of *infinitary first-order combinatorics*. It is important that the pointed out classes of transformations of theories preserve definite semantic layers of model-theoretic properties. Methods of finitary first-order combinatorics preserve a semantic layer called *finitary*, while methods of infinitary first-order combinatorics preserve a semantic layer called *infinitary*.

Работы [11], [12], и [14] представляют концептуальную основу финитарной комбинаторики первого порядка. Главной целью этого подхода является характеризация структуры алгебр Тарского-Линденбаума исчислений предикатов конечных богатых сигнатур. В настоящей работе приведены ключевые технические спецификации для комбинаторного подхода первого порядка. Определён статус процедур редукции сигнатур и универсальной конструкции конечно аксиоматизируемых теорий в рамках комбинаторного подхода первого порядка.

Preliminaries

We consider theories in *first-order predicate logic* with *equality* and use general concepts of model theory, algorithm theory, constructive models and Boolean algebras that can be found in [4], [16], and [1]. Generally, *incomplete* theories are considered. In this work, we consider just signatures admitting Gödel's numberings of formulas. Such a signature is called *enumerable.*

In description of a signature, capital letters are used for predicate symbols, while lowercase letters for function and constant symbols. Moreover, superscripts mean arities of corresponding symbols. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0,\ldots,x_{k-1}$, $SL(\sigma)$ is the set of all sentences of signature σ. By *GR*, we denote the Graph theory of signature $\sigma_{GR}=\{\Gamma^2\}$ defined by axioms $(\forall x)\neg\Gamma(x,x)$ and $(\forall x)(\forall y)[\Gamma(x,y)\leftrightarrow\Gamma(y,x)]$, while *GRE* denotes an extension of *GR* defined by extra axioms $(\exists x,y)\Gamma(x,y)$ and $(\exists x,y)\big[(x\neq y)\,\&\,\neg\Gamma(x,y)\big]$. For a set of natural numbers, c.e. means *computably enumerable.* For a theory, c.a. means *computably axiomatizable*, while f.a. means *finitely axiomatizable.*

A finite signature is called *rich* if it contains at least one n-ary predicate or function symbol for $n\geqslant 2$, or two symbols of unary functions. Consider two signatures σ_1 and σ_2. We say that σ_1 *is covered* by σ_2, symbolically $\sigma_1\leqslant\sigma_2$, if there is a mapping $\lambda:\sigma_1\to\sigma_2$ such that the following conditions are satisfied for all $\mathfrak{s}\in\sigma_1$: (a) $\mathfrak{s}$ and $\lambda(\mathfrak{s})$ are symbols of the same type (either predicates, or functions, or constants); (b) arity of $\mathfrak{s}\leqslant$ arity of $\lambda(\mathfrak{s})$ in all cases when $\mathfrak{s}$ is a predicate or function symbol. By definition, for any finite signature σ the following relation is satisfied:

$$\sigma \textit{ is rich} \Leftrightarrow \{P^2\}\leqslant\sigma \textit{ or } \{f^1,h^1\}\leqslant\sigma \textit{ or } \{g^2\}\leqslant\sigma. \tag{0.1}$$

We say that formula $\theta(x)$ presents a distinguished element (constant) in T if sentences $(\exists x)\theta(x)$ and $(\forall y)(\forall z)[\theta(y)\,\&\,\theta(z)\to(y=z)]$ are provable in T. The formula $\theta(x)$ points out an element that can be used for trivial values of new constants in the theory.

For a constant symbol c that is not in σ, statement "all σ-symbols are defined c-trivially" means the following set of formulas:

$$\begin{aligned}&\texttt{(a)}\ (\forall x_1\ldots x_n)\neg P(x_1,\ldots,x_n),\ P^n\in\sigma,\\&\texttt{(b)}\ (\forall x_1\ldots x_m)\big(f(x_1,\ldots,x_m)=x_1\big),\ f^m\in\sigma,\\&\texttt{(c)}\ a=c,\ \textit{for each constant symbol}\ a\in\sigma.\end{aligned}\tag{0.2}$$

For a signature σ and a unary predicate $U^1\notin\sigma$, statement "all σ-symbols are defined trivially outside $U(x)$" means the following set of formulas:

$$\begin{aligned}&\texttt{(a)}\ \neg U(x_i)\to\neg P(x_1,\ldots,x_i,\ldots,x_n),\ P^n\in\sigma,\ 1\leqslant i\leqslant n,\\&\texttt{(b)}\ \neg U(x_j)\to f(x_1,\ldots,x_j,\ldots,x_m)=x_1,\ f^m\in\sigma,1\leqslant j\leqslant m.\end{aligned}\tag{0.3}$$

Let σ be a signature and Σ a subset of $SL(\sigma)$. We denote by $[\Sigma]^\sigma$ a theory of signature σ generated by Σ as a set of its axioms. There is one more version of the definition. Let $\Sigma\subseteq SL(\sigma)$ be a set of sentences. We nenote by $[\Sigma]^\star$ a theory of signature $\sigma'\subseteq\sigma$ generated by Σ as a set of its axioms, where σ' contains only those symbols from σ which occur in formulas of Σ. By σ^∞, we denote a fixed (very large) enumerable signature containing countably many constant symbols, symbols of propositional variables, and predicate and function symbols of each arity $n\geqslant 1$. It is supposed that each considered signature σ is a part of the universal signature σ^∞. By $\mathfrak{S}$, we denote the set of all possible enumerable signatures $\sigma\subseteq\sigma^\infty$. We use a fixed Gödel's numbering Φ_k, $k\in\mathbb{N}$, for the set of sentences of a fixed signature σ, and Gödel's numbering Φ_k^∞, $k\in\mathbb{N}$, for the set of sentences of the universal enumerable signature σ^∞.

Using Post's numbering W_n, $n\in\mathbb{N}$, for the family of all computably enumerable sets, we organize an effective numbering for the class of all computably axiomatizable theories. Two versions of indices are possible. The first one presents c.e. indices of c.a. theories of an enumerable or finite signature σ. If a theory T of signature σ is defined by set of axioms $\{\Phi_i \mid i\in W_m\}$, the number m is called a *computably enumerable index* or simply *c.e. index* of T. The second version represents weak indices for theories of different enumerable signatures $\sigma\subseteq\sigma^\infty$. For a given $m\in\mathbb{N}$, we consider a set of axioms $\Sigma=\{\Phi_i^\infty\mid i\in W_m\}$ and construct theory $T=[\Sigma]^\star$. The number m is called a *weak computably enumerable index* or simply *weak c.e. index* of the theory T. As for finitely axiomatizable theories, any such theory F is defined by a finite system A of axioms and therefore, by a single formula Φ which is a conjunction of formulas from A. If a f.a. theory F of signature σ is defined by an axiom Φ_m, the number m is called Gödel's number or simply *strong index* of F. For a given $m\in\mathbb{N}$, we consider an f.a. theory $F=[\Phi_m]^\star$. This number m is called a *universal Gödel's number* or simply *universal strong index* of the theory F. By $T^\sigma\{n\}$ we denote a theory of signature σ with c.e. index n, while $T^\star\{n\}$ is a theory with weak c.e. index n. Furthermore, by $F^\sigma\{n\}$ we denote a f.a. theory of signature σ with Gödel's number n, while $F^\star\{n\}$ is an f.a. theory with weak strong index n.

1 Finitary combinatorics and the finite signature reduction procedure

By *ACL*, we denote the layer of model-theoretic properties preserved by all possible Cartesian interpretations between computably axiomatizable theories. It is called the *Cartesian* semantic layer; alternatively, it is referred to as a *working release* of the *finitary semantic layer*. Detailed definitions concerning the layer *ACL*, as well as the other concepts of finitary first-order combinatorics, are given in [15].

The following common statement takes place:

Theorem 1.1. [Finite signature reduction procedure: a standard form] *It is possible to determine in a regular way the operator of the following form*

$$\mathrm{Redu}: T\mathfrak{S}^{\phi} \times \sigma FinRich \;\to\; T\mathfrak{S}^{\phi} \times ICartes^{\phi},$$

called the finite signature reduction procedure, where $\sigma FinRich$ is the set of all finite rich signatures $\sigma \subseteq \sigma^{\infty}$, $T\mathfrak{S}^{\phi}$ is the set of all theories of any finite signatures $\sigma \subseteq \sigma^{\infty}$, and $ICartes^{\phi}$ is the set of all Cartesian interpretations between theories of finite signatures. Moreover, all requirements listed below are satisfied.

Let T be a theory of a finite signature τ and σ be an arbitrary finite rich signature. Applying the mapping Redu *we obtain*

$$\mathrm{Redu}(T,\sigma) = (S,I),$$

where S is a theory of signature σ, while I is an interpretation of T in S, such that the following assertions are satisfied:

`Reference_Block` (1.1)

(a) *I is an $\exists\cap\forall$-presentable Cartesian interpretation of theories (thereby, the interpretation I defines a computable isomorphism $\mu : \mathcal{L}(T) \to \mathcal{L}(S)$ preserving model-theoretic properties of the semantic layer ACL*),

(b) *T is c.a. $\Leftrightarrow$ S is c.a.; in the case when T is a c.a. theory, c.e. indices of both S and I are found effectively in a pair of parameters consisting of a c.e. index of the input theory T and a Gödel number of the target finite rich signature σ,*

(c) *T is f.a. $\Leftrightarrow$ S is f.a.; in the case when T is a f.a. theory, both a Gödel number of S and a c.e. index of I are found effectively in a pair of parameters consisting of Gödel numbers of the input theory T and the target finite rich signature σ.*

`End_Ref`

Proof of Theorem 1.1 is given [15].

By specification, starting from a pair of input parameters (T,σ), the procedure Redu yields a theory S together with an interpretation I defining a computable isomorphism $\mu:\mathcal{L}(T)\to\mathcal{L}(S)$ between the Tarski-Lindenbaum algebras. A simpler record $S=\mathrm{Redu}(T,\sigma)$ is also possible, assuming that the interpretation I and isomorphism μ are omitted in the context.

Now, we pass to some generalization of the signintre reduction statement. It represents a special method of reduction of finitely axiomatizable theories to the graph theory *GRE* controlled with a level parameter.

The following technical statement takes place:

Lemma 1.2. [Parameterized Signature Reduction Statement] *Let $\sigma_{GR}=\{\Gamma^2\}$ be the signature of the graph theory GRE. There exist an effective sequence of sentences*

$$\theta_k,\ k\in\mathbb{N}\setminus\{0,1\},$$

of the signature σ_{GR} and a procedure RedLev *with two parameters*

$$\mathrm{RedLev}(e,T),\ e\in\mathbb{N},\ T \textit{ is a theory of a finite signature},$$

satisfying the following properties. Given an integer parameter $e\geqslant 2$ (called the level parameter) and a finitely axiomatizable theory F of a finite signature σ. Effectively in e and F, a pair of objects

$$(H,I)=\mathrm{RedLev}(e,F)$$

is constructed by the procedure RedLev *of the following form:*

H is a finitely axiomatizable theory of signature $\sigma_{GR}=\{\Gamma^2\}$,
I is a Cartesian interpretation of F in H.

Particularly, I determines a computable isomorphism $\mu:\mathcal{L}(F)\to\mathcal{L}(H)$ preserving model-theoretic properties of Cartesian semantic layer ACL (the more, any smaller layer $L\subseteq ACL$).

Moreover, the following assertions hold:

`Reference_Block` (1.2)

(a) *$GRE\vdash\theta_{k+1}\to\theta_k$, for all $k\in\mathbb{N}\setminus\{0,1\}$,*
(b) *H is an extension of the theory $GRE_e=GRE\cup\{\theta_e\}$,*
(c) *$H\vdash\theta_e\,\&\,\neg\theta_{e+1}$,*
(d) *$GRE_\omega=GRE\cup\{\theta_2,\theta_3,..,\theta_k,...\ ;\ k\in\mathbb{N}\setminus\{0,1\}\}$ is a complete decidable theory without finite models.*

`End_Ref`

Proof. Use construction used in main stage fP-to-Graph in the proof for signature reduction procedure in Theorem 1.1. It is possible to modify forms of coding configurations such that they become depending on the level parameter e, cf. Section 5 in [8], or Section 4.2 in [10]. □

2 Infinitary combinatorics and status of the universal construction

By *MQL*, we denote the *infinitary* semantic layer consisting of model-theoretic properties that are preserved by the class of quasiexact interpretations between c.e. theories. The layer *MQL* is controlled by a standard version $\mathbb{F}\mathbb{U}$ of the universal construction of finitely axiomatizable theories, cf. [9, Ch. 6].

Description of the construction $\mathbb{F}\mathbb{U}$ represents a sophisticated text that turns out to be difficult for reading and understanding. There are some weaker versions of the universal construction with a simplified or even omitted rigidity mechanism, thus, controlling smaller layers of model-theoretic properties. However, the difficulty of studying these constructions is practically the same as in the case of construction $\mathbb{F}\mathbb{U}$. There is even a weaker version $\mathbb{F}\mathbb{C}^{\circ}$ of the universal construction that is described in [12]. Construction $\mathbb{F}\mathbb{C}^{\circ}$ is obtained as a routine corollary of the canonical construction presented in [9, Ch. 3]. Therefore, $\mathbb{F}\mathbb{C}^{\circ}$ is said to be the *canonical-mini* construction, or *universal-under-canonical* construction. Canonical-mini construction has a standard formulation of the universal construction supporting a relatively small layer of model-theoretic properties. At the same time, the canonical-mini construction is significantly easier to understand than any normal version of the universal construction. Moreover, while studying canonical-mini construction no necessity to be familiar with the technically complicated definition of a quasiexact interpretation. On the other hand, a particular method of compact binary trees is required for the canonical-mini construction, while any normal version of the universal construction does without this method. Finally, we notice that the Hanf construction $\mathbb{H}$ (cf. either [2, Th.1] or [9, Sec. 6.1]) can also be considered as a (weakest) release of the universal construction controlling an empty semantic layer of model-theoretic properties.

In this paper, we suppose that a fixed release of the universal construction is accepted, denoted by $\mathbb{U}$, that can control a sublayer

$$\mathit{MQL} \subseteq \mathit{MQL} \tag{2.1}$$

of the infinitary layer *MQL*. Moreover, we also suppose that the construction $\mathbb{U}$ is given by its *primitive form* $\hat{\mathbb{U}}$ without the effectiveness requirement in the passage from an input computably axiomatizable theory to the target finitely axiomatizable theory. In the paper, statements depending on the universal construction are marked with [$\mathbb{U}$].

The pointed out primitive form $\hat{\mathbb{U}}$ of the construction $\mathbb{U}$ is presented by:

Statement 2.1. [Generic universal construction: a primitive form] *The following assertion holds for the sublayer MQL of the layer MQL:*

$$(\forall\ \textit{c.a. theory}\ T)(\exists\ \textit{f.a. theory}\ F)\ \big[\,T \equiv_{MQL} F\,\big], \tag{2.2}$$

where $T \equiv_{MQL} F$ *means that there is a computable isomorphism* $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ *between the Tarski-Lindenbaum algebras preserving all model-theoretic properties within the pointed out layer MQL.*

A more common *normal* formulation of the universal construction $\mathbb{U}$:

Statement 2.2. [GENERIC UNIVERSAL CONSTRUCTION: A NORMAL FORM] *Given an arbitrary computably axiomatizable theory* T *and a finite rich signature* σ*. Effectively in a weak c.e. index of* T *and a Gödel number of* σ*, one can construct a finitely axiomatizable theory* $F = \mathbb{U}(T,\sigma)$ *of signature* σ *together with a computable isomorphism* $\mu : \mathcal{L}(T) \to \mathcal{L}(F)$ *between the Tarski-Lindenbaum algebras preserving all model-theoretic properties within the layer MQL* $\subseteq$ *MQL.*

The case *MQL*$=\varnothing$ in Statement 2.1, as well as in Statement 2.2, corresponds to the *Hanf construction* (called earlier as *Hanf's Localized Statement*), [2]. Obviously, the Hanf construction is a particular case of the universal construction corresponding to the case of an empty layer of the controlled model-theoretic properties.

Lemma 2.3. *The following assertions hold*:

(a) *For a fixed semantic layer MQL* $\subseteq$ *MQL, primitive form* (2.2) *of the universal construction is an immediate consequence of its normal form with this layer MQL.*

(b) *Having any version of the universal construction in a primitive form* (2.2) *that controls a semantic layer MQL* $\subseteq$ *MQL, we can deduce a normal form of the universal construction presented in Statement* 2.2 *with this layer MQL, restoring by that the missing effectiveness requirement, as well as the possibility to have a given finite rich signature.*

PROOF. Part (a) is obvious, while proof of Part (b) is given in Section 4.

Earlier, we fixed a sublayer (2.1) of the infinitary semantic layer that is controlled by a primitive form (2.2) of the universal construction. Having this convention accepted, by virtue of Lemma 2.3, we will also have Statement 2.2 presenting a normal version of the universal construction with the same layer (2.1).

3 Properties of effective numberings of the classes of theories

As a result, we have introduced c.e. indices and Gödel numbers for the following classes of theories:

`Reference_Block` (3.1)

`(a)` $F^{\sigma}{\{k\}}$, $k \in \mathbb{N}$, *the set of all f.a. theories of a fixed finite signature* σ,

`(b)` $F^{\star}{\{k\}}$, $k \in \mathbb{N}$, *the set of all f.a. theories of all possible finite signatures,*

`(c)` $T^{\tau}{\{k\}}$, $k \in \mathbb{N}$, *the set of all c.a. theories of a fixed enumerable or finite* *signature* τ,

(d) $T^{\star}{\{k\}}$, $k\in\mathbb{N}$, *the set of all c.a. theories of all possible enumerable signatures.*

End_Ref

Lemma 3.1. *The following assertions hold:*

(a) *collection* (3.1)(a) *represents a computable sequence of all possible, up to an algebraic isomorphism, finitely axiomatizable theories of a fixed finite signature* σ.

(b) *collection* (3.1)(b) *represents a computable sequence of all possible, up to an algebraic isomorphism, finitely axiomatizable theories of arbitrary finite signatures.*

PROOF. Immediately. □

Lemma 3.2. *The following assertions hold:*

(a) *collection* (3.1)(c) *represents a computable sequence of all possible, up to an algebraic isomorphism, computably axiomatizable theories of a fixed enumerable or finite signature* τ.

(b) *collection* (3.1)(d) *represents a computable sequence of all possible, up to an algebraic isomorphism, computably axiomatizable theories of arbitrary enumerable signatures.*

PROOF. Immediately. □

An effective reduction of Gödel numbers to other types of indices is possible.

Lemma 3.3. *There are general computable functions* $f_1(\sigma,x)$, $f_2(\tau,\sigma,x)$, *and* $f_3(x)$ *with* $\tau\in\sigma Enum$, $\sigma\in\sigma Fin$, *and* $x\in\mathbb{N}$, *such that we have for all* $n\in\mathbb{N}$:

(a) $F^{\sigma}{\{n\}}\approx_a F^{\star}{\{f_1(\sigma,n)\}}$,

(b) $F^{\sigma}{\{n\}}\approx_a T^{\tau}{\{f_2(\tau,\sigma,n)\}}$, *whenever* $\sigma\leqslant\tau$ (*see definition for* $\leqslant$ *in Preliminaries*),

(c) $F^{\star}{\{n\}}\approx_a T^{\star}{\{f_3(n)\}}$;

moreover, c.e. indices of algebraic isomorphisms in (a), (b), *and* (c) *are found effectively in* τ, σ, *and* n.

PROOF. By definition, finitely axiomatizable theory $F^{\sigma}{\{n\}}$ is defined by an axiom Φ_n. Effectively in σ and n one can find an integer m such that Φ^{∞}_m represents the same theory as an axiom in another system of numbering of theories. Thereby, we have obtained instructions for the function $f_1(\sigma,n)$. Parts (b) and (c) are proved by similar schemes.

Lemma 3.3 is proved. □

In this subsection, we consider some computational properties for the numberings of the classes of c.a. and f.a. theories introduced in (3.1).

Lemma 3.4. [EFFECTIVE CYLINDER PROPERTY] *Each of the numberings* (*a*), (*b*), (*c*), *and* (*d*) *in* (3.1) *is cylindric. More precisely, there are total* *computable functions* $f_k(n,i)$, $g_k(n,i,x)$, $k=1,2,3,4$, *satisfying for all* n, i:

$$f_k(n,0)=n,\ f_k(n,i)<f_k(n,i+1),\ k=1,2,3,4,$$

such that, the following properties are held for all $n,i\in\mathbb{N}$ *and all signatures* $\tau\in\sigma Enum$ *and* $\sigma\in\sigma FinRich$:

(a) $F^\sigma{}_{\{n\}}\approx_a F^\sigma{}_{\{f_1(n,i)\}}$; *moreover,* $\lambda x\, g_1(n,i,x)$ *represents this isomorphism,*

(b) $F^\star{}_{\{n\}}\approx_a F^\star{}_{\{f_2(n,i)\}}$; *moreover,* $\lambda x\, g_2(n,i,x)$ *represents this isomorphism,*

(c) $T^\tau{}_{\{n\}}\approx_a T^\tau{}_{\{f_3(n,i)\}}$; *moreover,* $\lambda x\, g_3(n,i,x)$ *represents this isomorphism,*

(d) $T^\star{}_{\{n\}}\approx_a T^\star{}_{\{f_4(n,i)\}}$; *moreover,* $\lambda x\, g_4(n,i,x)$ *represents this isomorphism.*

Proof. (a) Let σ be a finite rich signature and Φ_i, $i\in\mathbb{N}$, be a Gödel numbering of $SL(\sigma)$. By definition, formula Φ_n is an axiom of theory $F^\sigma{}_{\{n\}}$. Consider the following formula

$$\Phi^{(k)}=\Phi_n\,\&\,(\exists x)\big[\underbrace{(x=x)\,\&\,(x=x)\,\&\ldots\&\,(x=x)}_{k\ \ times}\big]. \tag{3.2}$$

Choose k such that Gödel number m of formula (3.2) satisfies $m>n$. Then, theory $F^\sigma{}_{\{m\}}$ defined by formula (3.2) as an axiom coincides with $F^\sigma{}_{\{n\}}$. Based on this construction, we obtain instructions for computing functions f_1 and g_1. Part (b) is proved by a similar method. Now, we pass to Part (c). By definition, $\{\Phi_i\mid i\in W_n\}$ is a set of axioms of theory $T^\sigma{}_{\{n\}}$. Based on the known properties of Post's numbering of c.e. sets, we can effectively find $m>n$ such that $W_m=W_n$, thus, theory $T^\sigma{}_{\{m\}}$ will coincide with $T^\sigma{}_{\{n\}}$. Based on this method, we can construct the functions required in (c). Part (d) can be proved by a similar method. □

In the following statement, for simplicity, we count that signatures τ and σ are parameters that are given by a c.e. index and, respectively, by a Gödel number.

Lemma 3.5. [Effective bijection property] *There are computable functions* p_i, f_i, $i=1,2,3$, *(whose arguments are seen below) satisfying the following properties for all* $\tau\in\sigma EnumRich$, $\sigma\in\sigma FinRich$, *and* $n,x\in\mathbb{N}$:

(a) $\lambda x\, p_1(\sigma,x):\mathbb{N}\to\mathbb{N}$ *is a permutation of* $\mathbb{N}$ *for all* σ,

(a′) $T^\star{}_{\{n\}}\equiv_{MQL}F^\sigma{}_{\{p_1(\sigma,n)\}}$; *moreover,* $\lambda x\, f_1(\sigma,n,x)$ *represents a computable isomorphism* $\mu:\mathcal{L}(T^\star{}_{\{n\}})\to\mathcal{L}(F^\sigma{}_{\{p_1(\sigma,n)\}})$ *of this similarity,*

(b) $\lambda x\, p_2(\tau,\sigma,x):\mathbb{N}\to\mathbb{N}$ *is a permutation of* $\mathbb{N}$ *for all* τ, σ,

(b′) $T^\tau{}_{\{n\}}\equiv_{MQL}F^\sigma{}_{\{p_2(\tau,\sigma,n)\}}$; *moreover,* $\lambda x\, f_2(\tau,\sigma,n,x)$ *represents a computable isomorphism* $\mu:\mathcal{L}(T^\tau{}_{\{n\}})\to\mathcal{L}(F^\sigma{}_{\{p_2(\tau,\sigma,n)\}})$ *of this similarity,*

(c) $\lambda x\, p_3(\sigma,x):\mathbb{N}\to\mathbb{N}$ *is a permutation of* $\mathbb{N}$ *for all* σ,

(c′) $F^\star{}_{\{n\}}\equiv_{ACL}F^\sigma{}_{\{p_3(\sigma,n)\}}$; *moreover,* $\lambda x\, f_3(\sigma,n,x)$ *represents a computable isomorphism* $\mu:\mathcal{L}(F^\star{}_{\{n\}})\to\mathcal{L}(F^\sigma{}_{\{p_3(\sigma,n)\}})$ *of this similarity.*

PROOF. (a) A passage from $F^{\sigma}{}_{\{n\}}$ to $T^{\star}{}_{\{m\}}$ is provided by Lemma 3.3(a,c). A back passage is ensured by an accepted version of the universal construction controlling semantic layer (2.1). Lemma 3.4 provides cylindric properties for these numberings. By applying method of proof of the known Myhill Theorem in algorithm theory, [16, Sec. 7.4], we will construct the demanded permutation. The obtained permutation preserves semantic layer MQL because we have applied Statement 2.2 of the universal construction.

(b) A passage from $F^{\sigma}{}_{\{n\}}$ to $T^{\tau}{}_{\{m\}}$ is provided by Lemma 3.3(b); possible, finite signature reduction procedure should be used in addition. A back passage is ensured by an accepted version of the universal construction. Based on cylindric properties, by applying Myhill's method, we can construct the required permutation. The obtained permutation preserves semantic layer MQL because we have applied Statement 2.2 of the universal construction.

(c) A passage from $F^{\sigma}{}_{\{n\}}$ to $F^{\star}{}_{\{m\}}$ is provided by Lemma 3.3(a). A back passage is ensured by the finite-to-finite signature reduction procedure. Based on cylindric properties, by applying Myhill's method, we can construct the required permutation. Obtained passage preserves semantic layer ACL because we have applied Theorem 1.1 in the transformation. □

4 Effectiveness of the universal construction

Now, we pass to the PROOF of Part (b) of Lemma 2.3.

First, we introduce an operation with a sequence of theories. We use sequence $T^{\star}{}_{\{n\}}$, $n\in\mathbb{N}$, including all, up to an algebraic isomorphism, c.a. theories, cf. (3.1)(d). Let $T^{\star}{}_{\{n\}}$ has signature σ_n. It is possible to assume that $\sigma_n \cap \sigma_k = \varnothing$ for all n, k such that $n \neq k$. Consider the following new signature

$$\sigma' = \{\mathcal{Z}_i^0 \mid i \in \mathbb{N}\} \cup \{U^1, c\} \cup \sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_k \cup \ldots,$$

where $\mathcal{Z}_i^0, i \in \mathbb{N}$, are symbols of nulary predicates (propositional variables). It is assumed that the symbols U, c, and $\mathcal{Z}_i$, $i \in \mathbb{N}$, do not belong to $\sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_k \cup \ldots$.

We are going to construct a theory $T^u_{c.a.}$ of signature σ' called a *simplest direct product of the sequence* $T^{\star}{}_{\{n\}}$, $n\in\mathbb{N}$, denoted

$$T^u_{c.a.} = \breve{\bigotimes}^{[EQ]}_{n\in\mathbb{N}} T^{\star}{}_{\{n\}}. \tag{4.1}$$

For comparison purposes, we use an alternative notation $T^{EQ}_{c.a.}$ for this theory. Theory $T^u_{c.a.}$ is defined by the following set of axioms (Ax-0):

1°. $U(x) \leftrightarrow (x \neq c)$,

2°. $(\exists x) U(x)$,

3°. $\mathcal{Z}_n \rightarrow \neg \mathcal{Z}_k$, $n, k \in \mathbb{N}$, $n \neq k$,

4°. $\mathcal{Z}_n \rightarrow$ (*all axioms of* $T^{\star}{}_{\{n\}}$ *are satisfied on* $U(x)$), $n\in\mathbb{N}$, 5°. $\mathcal{Z}_n \to$ (*outside* $U(x)$, σ_n-*symbols are defined trivially*), $n \in \mathbb{N}$,

6°. $\neg \mathcal{Z}_k \to$ (*all* σ_k-*symbols are defined c-trivially*), $k \in \mathbb{N}$.

Mention that, the term "defined c-trivially" is explained in Preliminaries.

REMARK. Later (in a further article devoted the combinatorial approach) we are planning to describe a common version of the operation of a direct product of a sequence of theories. The version we present in (4.1) via (Ax-0) plays the role of the simplest realization of a natural idea to link a sequence of theories together. Mnemonic entry $\breve{\bigotimes}^{[EQ]}_{i\in\mathbb{N}}(...)$ for this operation can be applied to an arbitrary sequence of theories, although we are interested with just the case presented in (4.1).

Lemma 4.1. *The following assertions hold:*

(a) *theory* $T^u_{c.a.} = \breve{\bigotimes}^{[EQ]}_{n\in\mathbb{N}} T^{\star}{}_{\{n\}}$ *is computably axiomatizable*;

(b) *for any* $n \in \mathbb{N}$, *theory* $T^u_{c.a.} \cup \{\mathcal{Z}_n\}$ *is algebraically isomorphic to a singleton extension* $T^{\star}{}_{\{n\}}\langle c\rangle$ *of the theory* $T^{\star}{}_{\{n\}}$;

(c) *there is a computable isomorphism* $\mu_n : \mathcal{L}(T^{\star}{}_{\{n\}}) \to \mathcal{L}(T^u_{c.a.} \cup \{\mathcal{Z}_n\})$ *preserving model-theoretic properties within the semantic layer ASL.*

PROOF. Part (a) is a consequence of the fact that the sequence $T^{\star}{}_{\{n\}}$, $n \in \mathbb{N}$, is computable. Part (b) is checked immediately. Statement of Part (c) is a consequence of Part (b). $\square$

Now, we turn immediately to prove Part (b) of Lemma 2.3.

Applying primitive form (2.2) of the universal construction to the theory $T^u_{c.a.}$, we find a finitely axiomatizable theory $F^* = \hat{\mathbb{U}}(T^u_{c.a.})$ together with a computable isomorphism

$$\mu^* : \mathcal{L}(T^u_{c.a.}) \to \mathcal{L}(F^*) \tag{4.2}$$

preserving model-theoretic properties of the layer MQL. Denote $\hat{\mathcal{Z}}_i = \mu^*(\mathcal{Z}_i)$, $i \in \mathbb{N}$. The effectiveness requirement is obtained as an immediate consequence of the universality property for theory $T^u_{c.a.}$ stated in Parts (a)–(c) of Lemma 4.1. Namely, we have to perform the following chain of transformations:

$$T^{\star}{}_{\{n\}} \overset{\alpha}{\mapsto} T^u_{c.a.} + \{\mathcal{Z}_n\} \overset{\beta}{\mapsto} \underbrace{\hat{\mathbb{U}}(T^u_{c.a.}) + \{\hat{\mathcal{Z}}_n\}}_{S} \overset{\gamma}{\mapsto} \mathrm{Redu}(S,\sigma), \tag{4.3}$$

where $\mathrm{Redu}(S,\sigma)$ is an application of the finite-to-finite signature reduction procedure. By Lemma 4.1(c), passage α in chain (4.3) defines a computable isomorphism between the Tarski-Lindenbaum algebras preserving properties of semantic layer $ASL \supseteq MQL$, while passage γ, by Theorem 1.1, defines a computable isomorphism preserving semantic layer $ACL \supseteq MQL$. As for passage β, by construction, it is obtained by a restriction from isomorphism (4.2), thus, β also defines a computable isomorphism between corresponding Tarski-Lindenbaum algebras preserving the layer MQL. It is possible to check that all parts involved in the chain (4.3) are built effectively in n and

σ yielding a finitely axiomatizable theory fitting to formulation of Statement 2.2. Thereby, the summary transformation (4.3) can play the role of a normal form of the universal construction preserving the semantic layer *MQL*.

Part (b) of Lemma 2.3 is proved. □

Conclusion

In terms of the two levels of expressive possibilities of first-order logic, methods and results of [2] and [7] correspond to the infinitary level of expressiveness of first-order logic. On the other hand, methods and results of [3], [5], [8], and [6] correspond to the finitary level of expressiveness of the logic. These two groups of works are based on different approaches and both deserve to be studied independently, possibly, supplemented by their comparison and benchmarking.

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Перетятькин М.Г. ПРОЦЕДУРА РЕДУКЦИИ СИГНАТУР И УНИВЕРСАЛЬНАЯ КОНСТРУКЦИЯ В КАЧЕСТВЕ МЕТОДОВ ПРЕОБРАЗОВАНИЯ ТЕОРИЙ В КОМБИНАТОРНОМ ПОДХОДЕ ПЕРВОГО ПОРЯДКА

Работа развивает предложенный ранее комбинаторный подход первого порядка представляющий концептуальную основу для исследования структуры обобщённых алгебр Тарского-Линденбаума исчислений предикатов конечных богатых сигнатур над финитарным и инфинитарным семантическими слоями. В настоящей работе приведены ключевые технические спецификации определяющие статус процедур редукции сигнатур и универсальной конструкции конечно аксиоматизируемых теорий в рамках комбинаторного подхода первого порядка.