Semantic types of computably axiomatizable theories and operations on them in the framework of the first-order combinatorial approach

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Abstract

In the work, we consider the semantic types representing the structure of the generalized Tarski-Lindenbaum algebras of first-order theories. A number of natural operations on semantic types are defined, and their properties are investigated. Furthermore, corresponding operations on theories supporting those on semantic types are described. These results can be used as technical tools for the solution of problems of characterization of universal semantic types.

Keywords: *First-order logic; Tarski-Lindenbaum algebra; model theoretic property; semantic type of the theory; computable isomorphism.*

Results of the works [1] and [2] describe some special methods of constructing computable isomorphisms between the Tarski-Lindenbaum algebras of predicate calculi $PC(\sigma_1)$ and $PC(\sigma_2)$ of finite rich signatures σ_1 and σ_2. Finite-to-finite signature reduction procedure is involved in the main construction of the work [2]; it would be natural to call these transformations of theories as methods of *finitary first-order combinatorics*. On the other hand, an available version of the universal construction of finitely axiomatizable theories is involved in the proof of the main statement of the work [1]; thus, it would be natural to call these transformations of theories as methods of *infinitary first-order combinatorics*. It is important that the pointed out classes of transformations of theories preserve definite semantic layers of model-theoretic properties. Methods of finitary first-order combinatorics preserve a semantic layer called *finitary*, while methods of infinitary first-order combinatorics preserve a semantic layer called *infinitary*.

The works [3], [4], and [5] represent a conceptual framework of finitary first-order combinatorics. The main goal of this approach is characterization of the structure of the Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures. In this work, the concept of a semantic type is introduced and the most important properties of such types are studied within the context of finitary and infinitary first-order combinatorics.

Preliminaries

We consider theories in *first-order predicate logic* with *equality* and use general concepts of model theory, algorithm theory, constructive models and Boolean algebras that can be found in [6], [7], and [8]. Main technical concept used in this work are found in [9], while definitions concerning semantic layers within the first-order combinatorial approach are found in [10]. Generally, *incomplete* theories are considered. In this work, we consider just signatures admitting Gödel's numberings of formulas. Such a signature is called *enumerable*.

In description of a signature, capital letters are used for predicate symbols, while lowercase letters for function and constant symbols. Moreover, superscripts mean arities of corresponding symbols. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0,\dots,x_{k-1}$, $SL(\sigma)$ is the set of all sentences of signature σ. By *GR*, we denote the Graph theory of signature $\sigma_{GR}=\{\Gamma^2\}$ defined by axioms $(\forall x)\neg\Gamma(x,x)$ and $(\forall x)(\forall y)[\Gamma(x,y)\leftrightarrow\Gamma(y,x)]$, while *GRE* denotes an extension of *GR* defined by extra axioms $(\exists x,y)\Gamma(x,y)$ and $(\exists x,y)\big[(x\neq y)\,\&\,\neg\Gamma(x,y)\big]$. For a set of natural numbers, c.e. means *computably enumerable*. For a theory, c.a. means *computably axiomatizable*, while f.a. means *finitely axiomatizable*.

A finite signature is called *rich* if it contains at least one n-ary predicate or function symbol for $n\geqslant 2$, or two symbols of unary functions. Consider two signatures σ_1 and σ_2. Let σ be a signature and Σ a subset of $SL(\sigma)$. We denote by $[\Sigma]^\sigma$ a theory of signature σ generated by Σ as a set of its axioms. There is one more version of the definition. Let $\Sigma\subseteq SL(\sigma)$ be a set of sentences. We nenote by $[\Sigma]^\star$ a theory of signature $\sigma'\subseteq\sigma$ generated by Σ as a set of its axioms, where σ' contains only those symbols from σ which occur in formulas of Σ. By σ^∞, we denote a fixed (very large) enumerable signature containing countably many constant symbols, symbols of propositional variables, and predicate and function symbols of each arity $n\geqslant 1$. It is supposed that each considered signature σ is a part of the universal signature σ^∞. By $\mathfrak{S}$, we denote the set of all possible enumerable signatures $\sigma\subseteq\sigma^\infty$. We use a fixed Gödel's numbering Φ_k, $k\in\mathbb{N}$, for the set of sentences of a fixed signature σ, and Gödel's numbering Φ_k^∞, $k\in\mathbb{N}$, for the set of sentences of the universal enumerable signature σ^∞.

Using Post's numbering W_n, $n\in\mathbb{N}$, for the family of all computably enumerable sets, we organize an effective numbering for the class of all computably axiomatizable theories. Two versions of indices are possible. The first one presents c.e. indices of c.a. theories of an enumerable or finite signature σ. If a theory T of signature σ is defined by set of axioms $\{\Phi_i\,|\,i\in W_m\}$, the number m is called a *computably enumerable index* or simply *c.e. index* of T. The second version represents weak indices for theories of different enumerable signatures $\sigma\subseteq\sigma^\infty$. For a given $m\in\mathbb{N}$, we consider a set of axioms $\Sigma=\{\Phi_i^\infty\,|\,i\in W_m\}$ and construct theory $T=[\Sigma]^\star$. The number m is called a *weak computably enumerable index* or simply *weak c.e. index* of the theory T. As for finitely axiomatizable theories, any such theory F is defined by a finite system A of axioms and therefore, by a single formula Φ which is a conjunction of formulas from A. If a f.a. theory F of signature σ is defined by an axiom Φ_m, the number m is called Gödel's number or simply *strong index* of F. For a given $m\in\mathbb{N}$, we consider an f.a. theory $F=[\Phi_m]^\star$. This number m is called a *universal Gödel's number* or simply *universal strong index* of the theory F. By $T^\sigma{}_{\{n\}}$ we denote a theory of signature σ with c.e. index n, while $T^\star{}_{\{n\}}$ is a theory with weak c.e. index n. Furthermore, by $F^\sigma{}_{\{n\}}$ we denote a f.a. theory of signature σ with Gödel's number n, while $F^\star{}_{\{n\}}$ is an f.a. theory with weak strong index n.

1 Numerated Boolean algebras and indices

A Boolean algebra $\mathcal{B}$ together with a numeration $\nu\colon\mathbb{N}\xrightarrow{onto}|\mathcal{B}|$, denoted by $(\mathcal{B},\nu)$, is called a *numerated Boolean algebra* if its signature operations are uniformly presentable by general computable functions on the ν-numbers; i.e., if there are general computable functions $u(x,y)$, $v(x,y)$, and $w(x)$ that represent operations in the Boolean algebra as follows for all $m,n\in\mathbb{N}$:

$$\begin{aligned} &\nu(m)\cup\nu(n)=\nu(u(m,n)),\\ &\nu(m)\cap\nu(n)=\nu(v(m,n)),\\ &-\nu(m)=\nu(w(m)). \end{aligned} \tag{1.1}$$

A numerated Boolean algebra $(\mathcal{B},\nu)$ is called a *c.e. Boolean algebra* if the equality in $\mathcal{B}$ is a computably enumerable relation in this numeration ν; i.e., there is a relation $E(x,y)$ in Σ_1^0 such that for all $m,n\in\mathbb{N}$

$$\nu(m)=\nu(n)\Leftrightarrow E(m,n). \tag{1.2}$$

Two numerated Boolean algebras $(\mathcal{B}_1,\nu_1)$ and $(\mathcal{B}_2,\nu_2)$ are called *equivalent* or *computably isomorphic*, denoted $(\mathcal{B}_1,\nu_1)\cong(\mathcal{B}_2,\nu_2)$, if there is an isomorphism μ between $\mathcal{B}_1$ and $\mathcal{B}_2$ and two general computable functions $f(x)$ and $g(x)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{N} & \overset{f}{\underset{g}{\rightleftarrows}} & \mathbb{N} \\ {\scriptstyle\nu_1}\downarrow & & \downarrow{\scriptstyle\nu_2} \\ \mathcal{B}_1 & \xrightarrow{\ \mu\ } & \mathcal{B}_2 \end{array} \tag{1.3}$$

Note the following simple fact.

Lemma 1.1. *Let $(\mathcal{B}_1,\nu_1)$ and $(\mathcal{B}_2,\nu_2)$ be c.e. Boolean algebras and μ be an isomorphism from $\mathcal{B}_1$ to $\mathcal{B}_2$. For μ, to be a computable isomorphism from $(\mathcal{B}_1,\nu_1)$ to $(\mathcal{B}_2,\nu_2)$, it is sufficient that there exists either of the two general computable functions $f(x)$ or $g(x)$ shown in the diagram* (1.3)*; then, the existence of the other one is supplied automatically.*

PROOF. Suppose that a general computable function $f(x)$ indicated in the diagram exists. As a function $g(x)$ for the back passage, we can choose that defined by the following rule

$$g(x)=r\Big((\mu t)\big[''\nu_2\big(f(r(t))\big)=\nu_2(x)''\ \textit{is computed for}\ \leqslant t\ \textit{steps}\big]\Big),$$

where $r(x)$ is a Cantor function for numbering of pairs of natural numbers. The other case when $g(x)$ exists is considered similarly. □

Introduce indices for c.e. Boolean algebras

Let $\mathscr{B}$ be a countable atomless Boolean algebra. Fix a numeration δ of $\mathscr{B}$ and a sequence g_k, $k\in\mathbb{N}$, of elements in $\mathscr{B}$ such that the following conditions are satisfied:

$$\begin{aligned} &\text{(a)}\quad \mathscr{B}\ \textit{is a countable atomless Boolean algebra},\\ &\text{(b)}\quad (\mathscr{B},\delta)\ \textit{is a constructive algebra},\\ &\text{(c)}\quad g_i,\ i\in\mathbb{N},\ \ \textit{is an effective in}\ \delta\ \textit{sequence of free generators for}\ \mathscr{B}. \end{aligned} \tag{1.4}$$

We will use the algebra (1.4) as a tool for presentation Boolean algebras.

Lemma 1.2. *For any c.e. Boolean algebra* $(\mathcal{B},\nu)$, *there is a c.e. filter* $\mathcal{F}$ *in* $\mathscr{B}$ *such that* $(\mathcal{B},\nu)\cong(\mathscr{B}/_{\mathcal{F}},{}^{\delta}/_{\mathcal{F}})$.

PROOF. Consider a mapping $\mu':\{g_i \mid i\in\mathbb{N}\}\xrightarrow{onto}|\mathcal{B}|$ defined by $\mu'(g_n)=\nu(n)$, $n\in\mathbb{N}$. Since $\mathscr{B}$ is freely generated with the sequence g_i, $i\in\mathbb{N}$, the mapping μ' can be extended to a homomorphism $\mu:\mathscr{B}\xrightarrow{onto}\mathcal{B}$. By construction, the filter $\mathcal{F}=\{a\in\mathscr{B}\mid\mu(a)=\mathbf{1}\}$ is computably enumerable in the numeration δ and the following is satisfied $(\mathcal{B},\nu)\cong(\mathscr{B}/_{\mathcal{F}},{}^{\delta}/_{\mathcal{F}})$. □

By $\mathcal{F}[Z]$, we denote a filter generated by a set Z in a Boolean algebra. Introduce the following series of notations in the Boolean algebra (1.4):

$$(a)\ \mathfrak{F}_m=\mathcal{F}[\{\delta(k)\mid k\in W_m\}],\quad m\in\mathbb{N}, \qquad (1.5)$$
$$(b)\ (\mathcal{B}_{\{m\}},\delta_{\{m\}})=(\mathscr{B}/_{\mathfrak{F}_m},{}^{\delta}/_{\mathfrak{F}_m}),\ m\in\mathbb{N}.$$

Obviously, the filter $\mathcal{F}[Z]$ is computably enumerable if the set Z is computably enumerable in the numeration δ. This ensures that the sequence (1.5)(b) contains each, up to an isomorphism, c.e. Boolean algebra. Moreover, the sequence (1.5)(b) is computable. If a c.e. Boolean algebra $(\mathcal{B},\nu)$ is computably isomorphic to the algebra $(\mathcal{B}_{\{m\}},\delta_{\{m\}})$ pointed out in (1.5), the number m is called a *computably enumerable index* or simply *index* for $(\mathcal{B},\nu)$. By Lemma 1.2, any c.e. Boolean algebra $(\mathcal{B},\nu)$ has at least one index m. For the sake of simplicity, we will often use short notation $\mathcal{B}_{\{m\}}$ instead of $(\mathcal{B}_{\{m\}},\delta_{\{m\}})$ omitting the Gödel numbering $\delta_{\{m\}}$ in context.

Let us define indices for isomorphisms of numerated Boolean algebras.

Suppose that numerated Boolean algebras $(\mathcal{B}_1,\nu_1)$ and $(\mathcal{B}_2,\nu_2)$ are isomorphic. Natural number n is said to be an index of the isomorphism if functions $f(t)=\varphi_{l(n)}(t)$ and $g(t)=\varphi_{r(n)}(t)$ are total; moreover, they are fit to the diagram (1.3) with a suitable abstract isomorphism $\mu:\mathcal{B}\to\mathcal{B}$. It is obvious that such number n completely defines all components pointed out in (1.1) for the computable isomorphism.

In the case of c.e. Boolean algebras, an alternative definition is possible.

Let $(\mathcal{B}_1,\nu_1)$ and $(\mathcal{B}_2,\nu_2)$ be c.e. Boolean algebras. If $m\in\mathbb{N}$ is such that the function $f(t)=\varphi_m(t)$ is total, and there is an isomorphism $\mu:\mathcal{B}_1\to\mathcal{B}_2$ for which (1.3) is satisfied, we say that m is an *index* for this computable isomorphism. By Lemma 1.1, a function $g(x)$ for the back passage in (1.3) is uniquely determined by m; moreover, the isomorphism μ itself, if such exists, is also uniquely determined by this number m.

2 Operations on abstract Boolean algebras

Let $\mathcal{B}$ be a Boolean algebra and a be an element in $\mathcal{B}$. We denote by $\mathcal{B}[a]$ the Boolean algebra of subelements of this element a; i.e., the Boolean algebra on the set

$$[\mathbf{0},a]=\{x\mid\mathbf{0}\subseteq x\subseteq a\}$$

with the operations $\cup$, $\cap$, and $\mathbf{0}$ available in $\mathcal{B}$; moreover, a is considered as the unit element, while the relative complements in the interval $[\mathbf{0},a]$ of $\mathcal{B}$ is accepted as the complement operation (i.e., the complement in $\mathcal{B}[a]$ is determined by the rule $-x = a \smallsetminus x$). One can easily check that this algebra is indeed a Boolean algebra. It is called the *restriction of* $\mathcal{B}$ *on the element* a.

Now, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Boolean algebras. We define the *direct product* $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ of these algebras as follows. We put

$$|\mathcal{B}| = |\mathcal{B}_1| \times |\mathcal{B}_2| = \{(a,b) \,|\, a \in \mathcal{B}_1,\ b \in \mathcal{B}_2\}.$$

The signature operations are defined by the following rule:

$$(a,b) \cup (c,d) = (a \cup c, b \cup d), \quad (a,b) \cap (c,d) = (a \cap c, b \cap d),$$
$$-(a,b) = (-a,-b), \quad \mathbf{0} = (\mathbf{0}_1, \mathbf{0}_2), \quad \mathbf{1} = (\mathbf{1}_1, \mathbf{1}_2).$$

One can easily check that the obtained algebra is indeed a Boolean algebra.

Define one more operation. Let $\mathcal{B}_n$, $n \in \mathbb{N}$, be an arbitrary countable sequence of Boolean algebras. We construct some new Boolean algebra

$$\mathcal{B} = \bigotimes_{n \in \mathbb{N}} \mathcal{B}_n \tag{2.1}$$

as follows. We set

$$|\mathcal{B}| = \big\{(a_0, a_1, \ldots) \,|\, a_i \in \mathcal{B}_i,\ (\exists s)\big[(\forall k > s)(a_k = \mathbf{0}_k) \vee (\forall k > s)(a_k = \mathbf{1}_k)\big]\big\}.$$

The signature operations are defined on this set by the following natural rule:

$$\begin{aligned}
&(a_0, a_1, \ldots) \cup (b_0, b_1, \ldots) = (a_0 \cup b_0, a_1 \cup b_1, \ldots),\\
&(a_0, a_1, \ldots) \cap (b_0, b_1, \ldots) = (a_0 \cap b_0, a_1 \cap b_1, \ldots),\\
&-(a_0, a_1, \ldots) = (-a_0, -a_1, \ldots),\\
&\mathbf{0} = (\mathbf{0}_0, \mathbf{0}_1, \ldots), \quad \mathbf{1} = (\mathbf{1}_0, \mathbf{1}_1, \ldots).
\end{aligned}$$

One can easily check that this algebra is indeed a Boolean algebra. It is called the *direct product of the sequence of algebras* $\mathcal{B}_n$, $n \in \mathbb{N}$.

Point out main dependencies among the operations.

Lemma 2.1. *The following assertions hold*:

(a) $\mathcal{B}_1 \otimes \mathcal{B}_2 \cong \mathcal{B}_2 \otimes \mathcal{B}_1$,

(b) $\big(\mathcal{B}_1 \otimes \mathcal{B}_2\big) \otimes \mathcal{B}_3 \cong \mathcal{B}_1 \otimes \big(\mathcal{B}_2 \otimes \mathcal{B}_3\big)$,

(c) $\bigotimes_{n \in \mathbb{N}} \mathcal{B}_n \cong \bigotimes_{n \in \mathbb{N}} \mathcal{B}_{f(n)}$, *for an arbitrary permutation* $f(x)$ *of the set* $\mathbb{N}$,

(d) $\bigotimes_{n \in \mathbb{N}} \mathcal{B}_n \cong \mathcal{B}_0 \otimes \bigotimes_{n \in \mathbb{N}} \mathcal{B}_{n+1}$,

(e) $\mathcal{B} \cong \mathcal{B}[a] \otimes \mathcal{B}[-a]$, *for any* $a \in \mathcal{B}$.

Proof. Immediately, from elementary properties of Boolean algebras. □

Describe some particular ultrafilter in the direct product (2.1). Consider the following sets in this algebra $\mathcal{B}$:

$$\hat{\mathfrak{F}} = Filter\{-\mathbf{1}_i \,|\, i \in \mathbb{N}\}, \quad \hat{\mathfrak{I}} = \{-a \,|\, a \in \hat{\mathfrak{F}}\}, \tag{2.2}$$

where $\mathbf{1}_i = (\mathbf{0}_0, \ldots, \mathbf{0}_{i-1}, \mathbf{1}_i, \mathbf{0}_{i+1}, \ldots)$.

Establish main properties of the ideal and the filter.

Lemma 2.2. *The filter* $\hat{\mathfrak{F}}$ *is an ultrafilter of Boolean algebra* $\mathcal{B}$, *while* $\hat{\mathfrak{I}}$ *is a maximal ideal of* $\mathcal{B}$; *moreover, the following assertions hold:*

$$\begin{aligned}
&\text{(a)}\quad a \in \hat{\mathfrak{F}} \Leftrightarrow (\exists m < \omega)\big[a \supseteq -(\mathbf{1}_0 \cup \mathbf{1}_1 \cup \ldots \cup \mathbf{1}_m)\big], \qquad (2.3)\\
&\text{(b)}\quad a \in \hat{\mathfrak{I}} \Leftrightarrow (\exists m < \omega)\big[a \subseteq \mathbf{1}_0 \cup \mathbf{1}_1 \cup \ldots \cup \mathbf{1}_m\big],\\
&\text{(c)}\quad \hat{\mathfrak{F}} \cup \hat{\mathfrak{I}} = |\mathcal{B}|, \quad \hat{\mathfrak{F}} \cap \hat{\mathfrak{I}} = \varnothing.
\end{aligned}$$

PROOF. From elementary properties of Boolean algebras. □

Describe Stone spaces of Boolean algebras obtained by the operations.

Lemma 2.3. *For any Boolean algebras, the following assertions hold*:

(a) $\mathrm{St}(\mathcal{B}[a]) = \{\mathfrak{F} \in \mathrm{St}(\mathcal{B}) \mid a \in \mathfrak{F}\}$,

(b) $\mathrm{St}(\mathcal{B}/_{\mathcal{F}}) = \{\mathfrak{F} \in \mathrm{St}(\mathcal{B}) \mid \mathcal{F} \subseteq \mathfrak{F}\}$,

(c) $\mathrm{St}(\mathcal{B}_1 \otimes \mathcal{B}_2) = \mathrm{St}(\mathcal{B}_1) \cup \mathrm{St}(\mathcal{B}_2)$,

(d) $\mathrm{St}\big(\bigotimes_{i\in\mathbb{N}} \mathcal{B}_i\big) = \bigcup_{i\in\mathbb{N}} \mathrm{St}(\mathcal{B}_i) \cup \{\hat{\mathfrak{F}}\}$, *where* $\hat{\mathfrak{F}}$ *is a particular ultrafilter defined by the rule* (2.2).

PROOF. From elementary properties of Boolean algebras. □

3 Operations on numerated Boolean algebras

Let $(\mathcal{B},\nu)$ be a numerated Boolean algebra and a be an element of $\mathcal{B}$ given by its number n_0, i.e., $a = \nu(n_0)$. We denote by $(\mathcal{B},\nu)[a]$ the numerated Boolean algebra of the form $(\mathcal{B}',\nu')$ such that $\mathcal{B}' = \mathcal{B}[a]$ and ν' is a numeration of the Boolean algebra $\mathcal{B}[a]$ which is induced from ν by the rule

$$\nu'(n) = \nu(n) \cap \nu(n_0).$$

So defined numeration ν' is denoted by $\nu[a]$ or by $\nu[\nu(n_0)]$. Thereby, the introduced operation has the following form: $(\mathcal{B},\nu)[a] = (\mathcal{B}[a],\nu[a])$. Notice that the definition of the numeration $\nu[a]$ is dependent on the choice of the number n_0 for a.

Lemma 3.1. *Let* $(\mathcal{B},\nu)$ *be a c.e. Boolean algebra. The following claims hold*:

(a) *for an arbitrary element* $a \in \mathcal{B}$, $a = \nu(n_0)$, *the algebra* $(\mathcal{B},\nu)[a]$ *is a c.e. Boolean algebra*,

(b) *independently of the choice of the number* n_0 *for the element* a, *the numerated Boolean algebra* $(\mathcal{B}[a],\nu[a])$ *is uniquely determined up to an isomorphism of numerated algebras.*

PROOF. By immediate check. □

Now, let $\mathcal{F}$ be a filter in a Boolean algebra $\mathcal{B}$, and $\mathcal{J}$ be the ideal that is dual to $\mathcal{F}$. Introduce an equivalence relation $\sim_{\mathcal{F}}$ on $\mathcal{B}$ as follows:

$$a \sim_{\mathcal{F}} b \Leftrightarrow (a \smallsetminus b) \cup (b \smallsetminus a) \in \mathcal{J} \Leftrightarrow -\big((a \smallsetminus b) \cup (b \smallsetminus a)\big) \in \mathcal{F}.$$

Determine Boolean operations on the quotient set $|\mathcal{B}|/_{\sim_{\mathcal{F}}}$ via representatives in the quotient classes. As a result, we obtain a Boolean algebra

$$\mathcal{B}/_{\mathcal{F}} = (|\mathcal{B}|/_{\mathcal{F}}, \cup, \cap, -, \mathbf{0}, \mathbf{1}),$$

which is called the *quotient* algebra of $\mathcal{B}$ *modulo* the filter $\mathcal{F}$. At a whole, the quotient operation has the following form

$$(\mathcal{B},\nu)/_{\mathcal{F}} = (\mathcal{B}/_{\mathcal{F}}, \nu/_{\mathcal{F}}).$$

In particular, we have:

Lemma 3.2. *If* $\mathcal{F}$ *is a computably enumerable filter in a c.e. Boolean algebra* $(\mathcal{B},\nu)$, *then* $(\mathcal{B}/_{\mathcal{F}}, \nu/_{\mathcal{F}})$ *is also a c.e. algebra.*

PROOF. Immediately, from elementary properties of Boolean algebras. □

Some relation between the two operations.

Lemma 3.3. *Let* $(\mathcal{B},\nu)$ *be a c.e. Boolean algebra and* a *be an element in* $\mathcal{B}$*. Suppose that* $\mathcal{F}$ *is the principal filter of* $\mathcal{B}$ *generated by* a*; i.e.,* $\mathcal{F}=\{c\in\mathcal{B}\mid -a\subseteq c\}$*. Then, we have* $(\mathcal{B},\nu)/_{\mathcal{F}}\cong(\mathcal{B},\nu)[a]$.

PROOF. From elementary properties of Boolean algebras. □

Now, let $(\mathcal{B}_1,\nu_1)$ and $(\mathcal{B}_2,\nu_2)$ be numerated Boolean algebras. We define a new numerated Boolean algebra

$$(\mathcal{B},\nu)=(\mathcal{B}_1,\nu_1)\otimes(\mathcal{B}_2,\nu_2)$$

as follows. We set $\mathcal{B}=\mathcal{B}_1\otimes\mathcal{B}_2$, while the numeration ν is defined by the rule

$$\nu(n)=\big(\nu_1\big(l(n)\big),\nu_2\big(r(n)\big)\big),\ n\in\mathbb{N},$$

where $c(x,y)$, $l(z)$, and $r(z)$ are standard pair numeration functions. So defined numeration ν is called the *direct product* of the numerations ν_1 and ν_2 and is denoted by $\nu=\nu_1\otimes\nu_2$. Notice that the numeration $\nu_1\otimes\nu_2$, actually, depends on the choice of the pair numeration functions. However, an isomorphism type of the numerated Boolean algebra $(\mathcal{B}_1\otimes\mathcal{B}_2,\nu_1\otimes\nu_2)$ does not depend on the choice of the functions, provided that they are chosen among general computable functions.

The following property of the introduced operation holds.

Lemma 3.4. *Algebra* $(\mathcal{B},\nu)=(\mathcal{B}_1,\nu_1)\otimes(\mathcal{B}_2,\nu_2)$ *is a c.e. Boolean algebra if and only if both* $(\mathcal{B}_1,\nu_1)$ *and* $(\mathcal{B}_2,\nu_2)$ *are c.e. algebras.*

PROOF. By immediate check. □

At last, we consider an operation of the direct product of a sequence. Let $(\mathcal{B}_n,\nu_n)$, $n\in\mathbb{N}$, be a sequence of numerated Boolean algebras. We define a numerated algebra

$$(\mathcal{B},\nu)=\bigotimes\nolimits_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n)$$

as follows. We put $\mathcal{B}=\bigotimes_{n\in\mathbb{N}}\mathcal{B}_n$ and define a numeration ν by the rule

$$\nu(n)=\begin{cases}\big(\nu_0(i_0),\ldots,\nu_{t-1}(i_{t-1}),\mathbf{0},\mathbf{0},\ldots\big), & \text{if } n=2k,\ \varkappa(k)=\langle i_0,\ldots,i_{t-1}\rangle,\\ \big(\nu_0(i_0),\ldots,\nu_{t-1}(i_{t-1}),\mathbf{1},\mathbf{1},\ldots\big), & \text{if } n=2k+1,\ \varkappa(k)=\langle i_0,\ldots,i_{t-1}\rangle,\end{cases}$$

where $\varkappa$ is some Gödel numbering for the set of all finite tuples of natural numbers. We call such a numeration ν the *direct product of the sequence of numerations* ν_n, $n\in\mathbb{N}$, and denote by $\nu=\bigotimes_{n\in\mathbb{N}}\nu_n$.

Notice that the introduced operation $(\bigotimes_{n\in\mathbb{N}}\mathcal{B}_n,\bigotimes_{n\in\mathbb{N}}\nu_n)$, actually, depends on the accepted Gödel numbering for the tuples. Nevertheless, independently of the choice of the numbering $\varkappa$, this numerated Boolean algebra is uniquely determined up to a computable isomorphism.

The following property of the operation holds.

Lemma 3.5. *Direct product* $(\mathcal{B},\nu)=\bigotimes_{n\in\mathbb{N}}\big(\mathcal{B}_n,\nu_n\big)$ *is a c.e. Boolean algebra if and only if the sequence* $(\mathcal{B}_n,\nu_n)$*,* $n\in\mathbb{N}$*, is a computable sequence consisting of c.e. Boolean algebras.*

PROOF. By immediate check. □

Point out main dependencies among the operations.

Lemma 3.6. *For any numerated Boolean algebras, the following holds*:

(a) $(\mathcal{B}_1,\nu_1)\otimes(\mathcal{B}_2,\nu_2)\cong(\mathcal{B}_2,\nu_2)\otimes(\mathcal{B}_1,\nu_1)$,

(b) $\Big((\mathcal{B}_1,\nu_1)\otimes(\mathcal{B}_2,\nu_2)\Big)\otimes(\mathcal{B}_3,\nu_3)\cong(\mathcal{B}_1,\nu_1)\otimes\Big((\mathcal{B}_2,\nu_2)\otimes(\mathcal{B}_3,\nu_3)\Big)$,

(c) $\bigotimes_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n)\cong\bigotimes_{n\in\mathbb{N}}(\mathcal{B}_{f(n)},\nu_{f(n)})$, *for an arbitrary computable permutation $f(x)$ of the set $\mathbb{N}$,*

(d) $\bigotimes_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n)\cong(\mathcal{B}_0,\nu_0)\otimes\bigotimes_{n\in\mathbb{N}}(\mathcal{B}_{n+1},\nu_{n+1})$,

(e) $(\mathcal{B},\nu)\cong(\mathcal{B},\nu)[a]\otimes(\mathcal{B},\nu)[-a]$, *for any $a\in\mathcal{B}$.*

Proof. By immediate check. □

Mention an effectiveness property of the particular ultrafilter.

Lemma 3.7. *Consider the algebra $(\mathcal{B},\nu)=\bigotimes_{n\in\mathbb{N}}\big(\mathcal{B}_n,\nu_n\big)$ obtained from a computable sequence of c.e. Boolean algebras. Let ultrafilter $\hat{\mathfrak{F}}$ and maximal ideal $\hat{\mathfrak{I}}$ be defined in $(\mathcal{B},\nu)$ by the rule (2.2). The following assertions hold:*

(a) *$\hat{\mathfrak{F}}$ represents a computable subset of $\mathcal{B}$ in the numeration ν.*

(b) *$\hat{\mathfrak{I}}$ represents a computable subset of $\mathcal{B}$ in the numeration ν.*

Proof. By virtue of Lemma 2.2, both the ultrafilter $\hat{\mathfrak{F}}$ and its complement coincided with $\hat{\mathfrak{I}}$ are computably enumerable sets in the numeration ν. Thereby, they are computable sets. □

As a demonstration, we construct an algebra with the universality property.

Consider the sequence of numerated Boolean algebras (1.5)(b), which is computable and contains all, up to an isomorphism, c.e. Boolean algebras. Construct the direct product of the sequence:

$$(\mathscr{B}^*,\delta^*)=\bigotimes_{k\in\mathbb{N}}(\mathcal{B}_{\{k\}},\delta_{\{k\}}). \tag{3.1}$$

By Lemma 3.5, the algebra (3.1) is a c.e. Boolean algebra. On the other hand, the algebra (3.1) obviously satisfies the following universality property: for any c.e. Boolean algebra $(\mathcal{B},\nu)$, there is an element $a\in|\mathfrak{B}^*|$ such that $(\mathcal{B},\nu)\cong(\mathscr{B}^*,\delta^*)[a]$; moreover, δ^*-number of a and an index for the isomorphism is found effectively in an index for the algebra $(\mathcal{B},\nu)$.

4 Semantic types of theories

From the point of view of a semantic layer L, any computably axiomatizable theory T can be characterized by a 3-tuple $\big(\mathcal{L}(T),\gamma,\xi\big)$, where $\big(\mathcal{L}(T),\gamma\big)$ is its Tarski-Lindenbaum algebra with a Gödel numbering, while ξ is a mapping from the Stone space $St(\mathcal{L}(T))$ into the power-set $\mathcal{P}(L)=\{K|K\subseteq L\}$ which is defined as follows: for any T' in $St(\mathcal{L}(T))$ that is a complete extension of T, we put

$$\xi(T')=\{p\in L\,|\,T' \textit{ has the property } p\}.$$

As a matter of fact, so defined 3-tuple $(\mathcal{L}(T),\gamma,\xi)$ represents a full abstract exposition of T in terms of the semantic layer L. We call this tuple *generalized Tarski-Lindenbaum algebra of the theory T under the semantic layer L*, or briefly the *Tarski-Lindenbaum L-algebra* of the theory T.

Generalizing the situation, we introduce a special class of objects to present isomorphism types of the Tarski-Lindenbaum L-algebras under a semantic layer L of model-theoretic properties. Namely, consider an arbitrary 3-tuple of the form

$\mathfrak{B}=(\mathcal{B},\nu,\xi)$, where $(\mathcal{B},\nu)$ is a c.e. Boolean algebra, while ξ is a mapping from Stone space $St(\mathcal{B})$ into the power-set $\mathcal{P}(L)$. So defined 3-tuple $(\mathcal{B},\nu,\xi)$ is called an *abstract semantic L-type*, or simply a *semantic type.*

Let $\mathfrak{B}_1=(\mathcal{B}_1,\nu_1,\xi_1)$ and $\mathfrak{B}_2=(\mathcal{B}_2,\nu_2,\xi_2)$ be two abstract semantic types under a semantic layer L. The types $\mathfrak{B}_1$ and $\mathfrak{B}_1$ are called *computably isomorphic* or *equivalent*, written $\mathfrak{B}_1\equiv_L\mathfrak{B}_2$, if there is a computable isomorphism $\mu:(\mathcal{B}_1,\nu_1)\to(\mathcal{B}_2,\nu_2)$ such that for any ultrafilter $\mathcal{F}_1\in\mathrm{St}(\mathcal{B}_1)$ and corresponding ultrafilter $\mathcal{F}_2\in\mathrm{St}(\mathcal{B}_2)$, $\mathcal{F}_2=\mu(\mathcal{F}_1)$, the following equality takes place:

$$\xi_1(\mathcal{F}_1)=\xi_2(\mathcal{F}_2). \tag{4.1}$$

Notice that computability of the isomorphism μ only requires that some general computable functions $f(x)$ and $g(x)$ exist for which the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{N} & \overset{f}{\underset{g}{\rightleftarrows}} & \mathbb{N} \\ {\scriptstyle\nu_1}\downarrow & & \downarrow{\scriptstyle\nu_2} \\ \mathcal{B}_1 & \xrightarrow{\ \mu\ } & \mathcal{B}_2 \end{array}$$

while an extra condition (4.1) is meant abstractly; i.e., none supporting effective method is supposed to verify the condition.

Let T be a theory and L be a semantic layer. Consider generalized Tarski-Lindenbaum algebra $(\mathcal{L}(T),\gamma,\xi)$ of T under the semantic layer L. If $\mathfrak{B}$ is a semantic type satisfying $(\mathcal{L}(T),\gamma,\xi)\equiv_L\mathfrak{B}$, we say that T *has the semantic type* $\mathfrak{B}$ *under* L, or that the *semantic type* $\mathfrak{B}$ *is presented* (*realized*) *in* T *under* L. By $€(T)$, we denote the semantic type of a theory T under the full semantic layer AL, while $€_L(T)$ stands for the semantic type of T under a semantic layer $L\subseteq AL$.

One can see that the concept of a semantic type together with the equivalence relation for such objects are in exact correspondence with the relation of semantic similarity of theories under a semantic layer.

Namely, the following statement takes place:

Lemma 4.1. *Let T and S be theories of enumerable signatures and L be a semantic layer. Then, the following assertions are equivalent:*

(a) *T and S are semantically similar under L,*

(b) $€(T)\equiv_L €(S)$.

PROOF. Immediately, from definitions. □

Introduce the following notations for classes of semantic types:

SemTypA(L), *is the set of all abstract semantic types under the layer* L,

SemTypE(L), *is the set of all computably enuberable types under* L,

SemTypF(L), *is the set of all finitely axiomatizable types under* L.

The class *SemTypE*(L) represents all those abstract semantic types which are realized in computably axiomatizable theories, while *SemTypF*(L) represents the types realized in finitely axiomatizable theories. If $\mathfrak{B}\in$ *SemTypF*(L), we use an alternative entry "$\mathfrak{B}$ is an *$\mathcal{F}$-type under the semantic layer L*" instead; similar terms "$\mathcal{E}$-type" and "$\mathcal{A}$-type" are also applicable to the classes *SemTypE*(L) and *SemTypA*(L).

Lemma 4.2. [$\mathbb{U}$] *Any $\mathcal{E}$-type under the layer MQL$\subseteq$MQL (controlled by an available version of the universal construction, cf.* [9, (2.1)]) *is an $\mathcal{F}$-type under MQL.*

Proof. Immediately, from [9, St. 2.1]. □

We need to develop effective versions of the statement of Lemma 4.2 with using a normal form of the universal construction presented in [9, St. 2.2]. First, we have to introduce concepts of indices for semantic types and isomorphisms between them.

5 Indices for semantic types and the isomorphism

We are in a position to introduce the concept of an *index* for computably axiomatizable and finitely axiomatizable types. Moreover, the concept of an index is also available for computable isomorphisms between the semantic types. Thus, semantic types become objects, which are involved in algorithmic constructions.

Turn to some more details.

Given a semantic layer $L\subseteq AL$. Let a semantic type $\mathfrak{B}\in SemTypE(L)$ be presented in computably axiomatizable theory $T^{\star}{}_{\{n\}}$ with an index n. In such case, the number n is called an *index* of this type $\mathfrak{B}$, symbolically $\mathfrak{B}=\mathcal{E}^{*}{}_{\{n\}}$. Similarly, if a type $\mathfrak{B}\in SemTypF(L)$ is presented in finitely axiomatizable theory $F^{\star}{}_{\{n\}}$ defined by a Gödel number n, the number n is called an *index* of this type $\mathfrak{B}$, symbolically $\mathfrak{B}=\mathcal{F}^{*}{}_{\{n\}}$.

Now, we define indices for isomorphisms of semantic types.

Let $\mathfrak{B}_1=(\mathcal{B}_1,\nu_1,\xi_1)$ and $\mathfrak{B}_2=(\mathcal{B}_2,\nu_2,\xi_2)$ be two abstract semantic types under a semantic layer L, and let μ be a computable isomorphism between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ for which the following diagram is commutative

$$\begin{array}{ccc} \mathbb{N} & \overset{f}{\underset{g}{\rightleftarrows}} & \mathbb{N} \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ \mathcal{B}_1 & \xrightarrow{\ \mu\ } & \mathcal{B}_2 \end{array}$$

with the function $f(x)=\lambda x\varphi_n(x)$ which should be total, where φ_n is nth function in Kleene's numbering of all partially computable functions, while $g(x)$ is a general computable function suitable for the back passage in accordance with the rule in Lemma 1.1. If so, the number n is called a c.e. *index* or simple *index* of the isomorphism μ. For this, we apply the following notation:

$$\overrightleftarrows{\varphi}_{\{n\}} \textit{ is the isomorphism with an index } n, \textit{ for } n\in\mathrm{Tot}.$$

Notice that the indices to isomorphisms of types belong to a complicated set Tot consisting of the Kleene indices to general computable functions. It is a known fact that this set is Π^0_2-complete. Nevertheless, the concept of an index for isomorphisms of types appears to be useful in various constructions.

We are going to point out a computably axiomatizable semantic type that satisfy a universality condition relative to the class of all such objects. Consider the following semantic type:

$$\mathcal{E}^{u}_{c.a.}=\text{\textit{£}}(T^{u}_{c.a.}), \tag{5.1}$$

where $T^{u}_{c.a.}$ is the computably axiomatizable theory defined in [9, (4.1)].

Lemma 5.1. *The following assertions take place for* $\mathcal{E}^u_{c.a}=₤(T^u_{c.a})$:

(a) *semantic type* $\mathcal{E}^u_{c.a.}$ *is computably axiomatizable under AL,*

(b) *semantic type* $\mathcal{E}^u_{c.a.}$ *has the following universality property under any semantic layer* $L\subseteq ASL$*: each computably axiomatizable type* $\mathfrak{B}$ *is equivalent to a segment* $\mathcal{E}^u_{c.a.}[a]$ *for some* $a\in|\mathcal{E}^u_{c.a.}|$*. Namely, the following explicit rule holds*:

$$\mathcal{E}^{\star}{}_{\{n\}}\equiv_L \mathcal{E}^u_{c.a.}[a],\ \ a \text{ is defined by } \mathcal{Z}_n \ (cf.\ \mathcal{Z}_n \text{ in } [9, Sec.4]);$$

moreover, both a Gödel number of the element a and an index of corresponding isomorphism are found effectively in n.

Proof. Immediately, from the properties of $T^u_{c.a.}$ stated in [9, Lem. 4.1]. □

6 Computable representative sequences of semantic types

Introduce the concept of computability for a family of semantic types.

Definition 6.A. A sequence $\mathfrak{B}_n$, $n\in\mathbb{N}$, of $\mathcal{F}$-types under a semantic layer L is called *computable* if there is a general computable function $f(x)$ such that $\mathfrak{B}_n\equiv_L \mathcal{F}^{\star}{}_{\{f(n)\}}$ for all $n\in\mathbb{N}$.

Definition 6.B. A sequence of $\mathcal{E}$-types $\mathfrak{B}_n$, $n\in\mathbb{N}$, under a semantic layer L is called *computable*, if there is a general computable function $f(x)$ such that $\mathfrak{B}_n\equiv_L \mathcal{E}^{\star}{}_{\{f(n)\}}$ for all $n\in\mathbb{N}$.

Give some natural concepts of representability of sequences.

Definition 6.C. A computable sequence of $\mathcal{F}$-types $\mathfrak{B}_n$, $n\in\mathbb{N}$, is called $\mathcal{F}$-*representative* under a semantic layer L if the following conditions are satisfied: (a) the sequence has effective cylindric properties, i.e., there are computable functions $f(n,k)$ and $p(n,k,x)$ satisfying $f(n,0)=n$, $f(n,k+1)>f(n,k)$, and $\mathcal{F}^{\star}{}_{\{n\}}\equiv_M \mathfrak{B}_{f(n,k)}$ for all $n,k\in\mathbb{N}$, moreover, the function $\lambda x p(n,k,x)$ represents this computable isomorphism; (b) the sequence presents effectively all possible $\mathcal{F}$-types under L, i.e., there are computable functions $h(n)$ and $q(n,x)$ such that $\mathfrak{B}_n\equiv_L \mathcal{F}^{\star}{}_{\{h(n)\}}$ for all $n\in\mathbb{N}$, moreover, the function $\lambda x q(n,x)$ represents this computable isomorphism.

Definition 6.D. A computable sequence of $\mathcal{E}$-types $\mathfrak{B}_n$, $n\in\mathbb{N}$, is called $\mathcal{E}$-*representative* under a semantic layer K if the following conditions are satisfied: (a) the sequence has effective cylindric properties, i.e., there are computable functions $f(n,k)$ and $p(n,k,x)$ satisfying $f(n,0)=n$, $f(n,k+1)>f(n,k)$, and $\mathcal{E}^{\star}{}_{\{n\}}\equiv_K \mathfrak{B}_{f(n,k)}$ for all $n,k\in\mathbb{N}$, moreover, the function $\lambda x p(n,k,x)$ represents this computable isomorphism; (b) the sequence presents effectively all possible $\mathcal{E}$-types under K, i.e., there are computable functions $h(n)$ and $q(n,x)$ such that $\mathfrak{B}_n\equiv_K \mathcal{E}^{\star}{}_{\{h(n)\}}$ for all $n\in\mathbb{N}$, moreover, the function $\lambda x q(n,x)$ represents this computable isomorphism.

Establish computability of the set of all finitely axiomatizable types.

Theorem 6.1. *The following assertions hold*:

(a) *The set of all finitely axiomatizable semantic types under an arbitrary semantic layer L, which are types of finitely axiomatizable theories of a fixed finite signature* σ*, is computable.*

(b) *The set of all finitely axiomatizable semantic types under an arbitrary semantic layer L, which are types of all possible finitely axiomatizable theories of any finite signatures, is computable.*

Proof. Consider the following sequences of semantic types:

$$\begin{aligned} &(\mathrm{a}) \quad \mathcal{F}^{\star}{}_{\{n\}} = \text{₤}(F^{\star}{}_{\{n\}}),\ n \in \mathbb{N}, \\ &(\mathrm{b}) \quad \mathcal{F}^{\sigma}{}_{\{n\}} = \text{₤}(F^{\sigma}{}_{\{n\}}),\ n \in \mathbb{N}. \end{aligned} \tag{6.1}$$

By applying [9, Lem. 3.1], we obtain exactly what is required. □

Similar statement concerning computably axiomatizable types.

Theorem 6.2. *The set of all computably axiomatizable semantic types under an arbitrary semantic layer K, which are types of all possible computably axiomatizable theories of any enumerable signatures, is computable.*

Proof. Consider the following sequences of semantic types:

$$\mathcal{E}^{\star}{}_{\{n\}} = \text{₤}(T^{\star}{}_{\{n\}}),\ n \in \mathbb{N}. \tag{6.2}$$

By applying [9, Lem. 3.2], we obtain exactly what is required. □

Theorem 6.3. *Given semantic layers L and K such that $L \subseteq ACL$ and $K \subseteq MSL$. The following assertions take place*:

(a) *there is a computable sequence of $\mathcal{F}$-types that is $\mathcal{F}$-representative under the layer L; in particular, both sequences* (6.1)*(a) and* (6.1)*(b) has these properties*;

(b) *any two computable $\mathcal{F}$-representative sequences $\mathcal{F}_i$, $i \in \mathbb{N}$, and $\mathcal{F}'_i$, $i \in \mathbb{N}$, of $\mathcal{F}$-types under the layer L are equivalent to each other; more precisely, there are computable functions $p(n)$ and $f(n,x)$, such that p is a permutation of the set $\mathbb{N}$ satisfying $\mathcal{F}_n \equiv_L \mathcal{F}'_{p(n)}$ for all $n \in \mathbb{N}$; moreover, the function $\lambda x f(n,x)$ represent an isomorphism for the pointed out similarity relation.*

(c) *there is a computable sequence of $\mathcal{E}$-types that is $\mathcal{E}$-representative under the layer K; in particular, the sequence* (6.2) *has these properties*;

(d) *any two computable $\mathcal{E}$-representative sequences $\mathcal{E}_i$, $i \in \mathbb{N}$, and $\mathcal{E}'_i$, $i \in \mathbb{N}$, of $\mathcal{E}$-types under the layer K are equivalent to each other; more precisely, there are computable functions $p(n)$ and $f(n,x)$, such that p is a permutation of the set $\mathbb{N}$ satisfying $\mathcal{E}_n \equiv_K \mathcal{E}'_{p(n)}$ for all $n \in \mathbb{N}$; moreover, the function $\lambda x f(n,x)$ represent an isomorphism for the pointed out similarity relation.*

Proof. Part (a) is proved by immediate checking Definition 6.C for semantic types of sequences of theories [9, (3.1)(a)], while for [9, (3.1)(b)], we have to use additionally a finite-to-finite signature reduction procedure, cf. [9, Th. 1.1]. Part (b) is established by applying method of proof of the Myhill Theorem in algorithm theory, [7, Sec. 7.4].

Part (c) is proved by immediate checking Definition 6.D for semantic types of the sequence of theories [9, (3.1)(d)]. Part (d) is established by applying method of proof of the Myhill Theorem, [7, Sec. 7.4]. □

The following statement represents an effective version of Lemma 4.2.

Theorem 6.4. [$\mathbb{U}$] *Given semantic layers L and K such that $L \subseteq ACL$ and $K \subseteq MSL$. Let $\mathcal{F}_i$, $i \in \mathbb{N}$, be a computable $\mathcal{F}$-representative sequence of $\mathcal{F}$-types under L and $\mathcal{E}_i$, $i \in \mathbb{N}$, a computable $\mathcal{E}$-representative sequence of $\mathcal{E}$-types under the layer K. There is a computable permutation $p: \mathbb{N} \to \mathbb{N}$ together with a computable function $h(n,x)$ such that $\mathcal{F}_i \equiv_{L \cap K \cap MQL} \mathcal{E}_{p(i)}$ for all $n \in \mathbb{N}$; moreover, the function $\lambda x h(n,x)$ represents an isomorphism for the pointed out similarity relation.*

Proof. By applying an available effective version of the universal construction, cf. [9, St. 2.2], we can build an effective embedding of sequence $\mathcal{F}_i$, $i \in \mathbb{N}$, into sequence $\mathcal{E}_{\{k\}}$, $k\in\mathbb{N}$; by Theorem 6.3, the former is isomorphic to the pointed out sequence $\mathcal{E}_i$, $i\in\mathbb{N}$. A back embedding can be established by a similar method with using [9, Lem. 3.5(a′)]. Finally, based on the cylindric properties of the sequences $\mathcal{F}_i$, $i\in\mathbb{N}$, and $\mathcal{E}_i$, $i\in\mathbb{N}$, stated in Lemma 6.3, we can build the demanded computable bijection by applying method of proof of the Myhill Theorem, [7, Sec. 7.4].

Theorem 6.4 is proved. □

Exercise 6.5. A computable sequence $\mathfrak{B}_i$, $i\in\mathbb{N}$, of finitely axiomatizable semantic types under $L\subseteq ACL$ can represent c.e. indices of $\mathcal{F}$-types if and only if it is $\mathcal{F}$-representative under L. Similarly, a computable sequence $\mathfrak{B}_i$, $i\in\mathbb{N}$, of computably axiomatizable types under $K\subseteq ASL$ can represent c.e. indices of $\mathcal{E}$-types if and only if it is $\mathcal{E}$-representative under K.

Hint. Use Theorem 6.3 together with definitions of indices for $\mathcal{F}$-types and $\mathcal{E}$-types, cf. Section 5.

7 Operations of restriction and factorization of semantic types

Consider a semantic type $\mathfrak{B}=(\mathcal{B},\nu,\xi)$ under a semantic layer L, and let $a=\nu[n_0]$ be an element in $\mathcal{B}$. Denote by $(\mathcal{B},\nu,\xi)[a]$ a semantic type $(\mathcal{B}',\nu',\xi')$, where $(\mathcal{B}',\nu')=(\mathcal{B}[a],\nu[a])$, while ξ' is the restriction of ξ up to Stone space of the algebra $\mathcal{B}[a]$ specified in Lemma 2.3 (a). So defined function ξ' is denoted by $\xi[a]$. Thus, the operation at a whole has the following form

$$(\mathcal{B},\nu,\xi)[a]=(\mathcal{B}[a],\nu[a],\xi[a]).$$

One can check that if the source type $\mathfrak{B}$ corresponds to a theory T, its restriction $\mathfrak{B}[a]$ corresponds to a finitely axiomatizable extension of the theory T.

In particular, we have:

Lemma 7.1. *Let L be a semantic layer. Consider a semantic type $\mathfrak{B}$ and an element $a\in|\mathfrak{B}|$. The following statements take place*:

(a) *if $\mathfrak{B}$ is a computably axiomatizable* (*the more, finitely axiomatizable*) *type, $\mathfrak{B}[a]$ is defined uniquely up to an isomorphism for any $a\in|\mathfrak{B}|$, independently of the number n_0 chosen for the element a,*

(b) *If $\mathfrak{B}\in SemTypF(L)$, then $\mathfrak{B}[a]\in SemTypF(L)$.*

(c) *If $\mathfrak{B}\in SemTypE(L)$, then $\mathfrak{B}[a]\in SemTypE(L)$,*

Proof. (a) Immediately; (b) from Lemma 3.1. □

One can note that, the operation of restriction by an element a is identical to the quotient operation modulo the principal filter generated by the element a. Thus, the operation of restriction by an element is a particular case of the more common quotient operation.

Turn to the quotient operation.

Consider a semantic type $\mathfrak{B}=(\mathcal{B},\nu,\xi)$, and let $\mathcal{F}$ be a filter of the Boolean algebra $\mathcal{B}$. Denote by ${}^{(\mathcal{B},\nu,\xi)}/_{\mathcal{F}}$ a semantic type of the form $(\mathcal{B}',\nu',\xi')$ where $(\mathcal{B}',\nu')$ is the quotient algebra of $(\mathcal{B},\nu)$ modulo $\mathcal{F}$, while the assignment function ξ' is defined by the following rule

$$\xi'(\mathcal{F}')=\xi(\mathcal{F}'),\quad \textit{for}\ \mathcal{F}'\in\mathrm{St}({}^{\mathcal{B}}/_{\mathcal{F}})=\{\mathcal{F}'\in\mathrm{St}(\mathcal{B})\mid\mathcal{F}\subseteq\mathcal{F}'\}.$$

So defined function ξ' is denoted by $\xi/_{\mathcal{F}}$. This is simply a restriction of ξ on the subspace of $\mathrm{St}(\mathcal{B})$ defined by the filter $\mathcal{F}$. Then, the quotient operation at a whole has the following form

$$(\mathcal{B},\nu,\xi)/_{\mathcal{F}}=(\mathcal{B}/_{\mathcal{F}},\nu/_{\mathcal{F}},\xi/_{\mathcal{F}}).$$

One can check that, if the source semantic type $\mathfrak{B}$ corresponds to a theory T, then the quotient type $\mathfrak{B}/_{\mathcal{F}}$ corresponds to an extension of the theory T defined by this filter $\mathcal{F}$ considered as an extra set of axioms.

In particular, we have:

Lemma 7.2. *Consider a semantic type $\mathfrak{B}\in SemTypE(L)$ under a semantic layer L, and let $\mathcal{F}$ be its computably enumerable filter. Then $\mathfrak{B}/_{\mathcal{F}}$ is in $SemTypE(L)$.*

PROOF. By immediate checking. □

Some relation between the two operations is described as follows:

Lemma 7.3. *Let $\mathfrak{B}=(\mathcal{B},\nu,\xi)$ be a semantic type under a semantic layer L and a be an element of $\mathcal{B}$. Suppose that $\mathcal{F}$ is a principal filter of $\mathcal{B}$ generated by a; i.e., $\mathcal{F}=\{c\in\mathcal{B}\mid a\subseteq c\}$. Then, we have $\mathfrak{B}/_{\mathcal{F}}\equiv_L\mathfrak{B}[a]$.*

PROOF. From elementary properties of Boolean algebras. □

8 Direct product of two semantic types

Let $\mathfrak{B}_1=(\mathcal{B}_1,\nu_1,\xi_1)$ and $\mathfrak{B}_2=(\mathcal{B}_2,\nu_2,\xi_2)$ be two semantic types under a semantic layer L. Define some new semantic type

$$\mathfrak{B}=(\mathcal{B},\nu,\xi)=(\mathcal{B}_1,\nu_1,\xi_1)\otimes(\mathcal{B}_2,\nu_2,\xi_2)=\mathfrak{B}_1\otimes\mathfrak{B}_2$$

under the same semantic layer L as follows. We let $(\mathcal{B},\nu)=(\mathcal{B}_1,\nu_1)\otimes(\mathcal{B}_2,\nu_2)$, while the assignment function ξ is determined by the rule

$$\xi(\mathcal{F})=\begin{cases}\xi_1(\mathcal{F}), \text{ if } \mathcal{F}\in\mathrm{St}(\mathcal{B}_1),\\ \xi_2(\mathcal{F}), \text{ if } \mathcal{F}\in\mathrm{St}(\mathcal{B}_2).\end{cases}$$

So defined operation $\mathfrak{B}_1\otimes\mathfrak{B}_2$ is called the *direct product* of semantic types $\mathfrak{B}_1$ and $\mathfrak{B}_2$, while the function ξ is called the *direct product* of functions ξ_1 and ξ_2, using for this an entry $\xi=\xi_1\otimes\xi_2$. The idea of the assignment function in the operation $\otimes$ is based on the following algebraic relation

$$\mathrm{St}(\mathcal{B}_1\otimes\mathcal{B}_2)=\mathrm{St}(\mathcal{B}_1)\cup\mathrm{St}(\mathcal{B}_2)$$

stated in Lemma 2.3(c). Thus, the two given assignment functions ξ_1 and ξ_2 defined on the disjoint parts $\mathrm{St}(\mathcal{B}_1)$ and $\mathrm{St}(\mathcal{B}_2)$ are simply assembled in one function $\xi=\xi_1\cup\xi_2$.

Now show that the introduced operation of product of two semantic types is supported by some natural operation on theories.

Let T_1 and T_2 be two theories of signatures σ_1 and σ_2. Consider the following signature

$$\sigma=\{Z_1^0,Z_2^0,U^1,c\}\cup\sigma_1\cup\sigma_2,$$

where Z_1 and Z_2 are nulary predicates. It is assumed that $\sigma_1\cap\sigma_2=\varnothing$; moreover, the new symbols Z_1, Z_2, U, and c are not included in $\sigma_1\cup\sigma_2$.

Construct a theory T of signature σ, denoted by $T_1 \otimes T_2$, which is defined by the following set of axioms (Ax-1) :

1°. $U(x) \leftrightarrow (x \neq c)$,

2°. $(\exists x) U(x)$,

3°. $Z_1 \leftrightarrow \neg Z_2$,

4°. $Z_1 \rightarrow$(*on* $U(x)$, *all axioms of the theory* T_1 *are satisfied*),

5°. $Z_1 \rightarrow$(*outside* $U(x)$, *all* σ_1*-symbols are defined trivially*),

6°. $Z_1 \rightarrow$(*all* σ_2*-symbols are defined c-trivially*),

7°. $Z_2 \rightarrow$(*on* $U(x)$, *all axioms of the theory* T_2 *are satisfied*),

8°. $Z_2 \rightarrow$(*outside* $U(x)$, *all* σ_2*-symbols are defined trivially*),

9°. $Z_2 \rightarrow$(*all* σ_1*-symbols are defined c-trivially*).

First, point out some model-theoretic meaning of the operation.

Lemma 8.1. *Let* T *be a theory of a signature* σ*, and* $\Theta \in SL(\sigma)$*. The following algebraic isomorphism takes place*:

$$T\langle c\rangle \approx_a \left[T \cup \{\Theta\}\right]^{\sigma} \otimes \left[T \cup \{\neg\Theta\}\right]^{\sigma}.$$

Proof. Immediately. We should mention that the constant c is necessary in the construction. It plays role on an available constant such that the demand "c-trivially" could be realized, cf. Preliminaries in [9]. □

Now, we state main properties of the operation.

Lemma 8.2. *The following statements take place:*

(a) *theory* $T_1 \otimes T_2$ *is finitely axiomatizable* $\Leftrightarrow$ *both theories* T_1 *and* T_2 *are finitely axiomatizable*,

(b) *theory* $T_1 \otimes T_2$ *is computably axiomatizable* $\Leftrightarrow$ *both theories* T_1 *and* T_2 *are computably axiomatizable*,

(c) $\notin(T_1 \otimes T_2) \equiv_{ASL} \notin(T_1) \otimes \notin(T_2)$,

(d) *under any semantic layer* $L \subseteq ASL$*, for any semantic types* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$*,* $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ *is finitely axiomatizable* $\Leftrightarrow$ *both* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are finitely axiomatizable.*

(e) *under any semantic layer* $L \subseteq ASL$*, for arbitrary semantic types* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$*,* $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ *is computably axiomatizable* $\Leftrightarrow$ *both* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are computably axiomatizable*,

Particularly, both statements (d) *and* (e) *are applicable to the semantic layer* $L = ACL$ *which is a sublayer of the semantic layer* ASL*.*

Proof. Immediately, from the construction. □

9 Direct product of a sequence of semantic types

We are in a position to specify a natural operation of the *direct product* of a *sequence of semantic types*. Let $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$, be a sequence of semantic types under a semantic layer L, and P be a complete theory of an enumerable signature that is used as an additional parameter in the operation.

Define a new semantic type

$$(\mathcal{B}, \nu, \xi) = \bigotimes_{n\in\mathbb{N}}^{[P]} \mathfrak{B}_n = \bigotimes_{n\in\mathbb{N}}^{[P]} (\mathcal{B}_n, \nu_n, \xi_n) \tag{9.1}$$

as follows. We set $(\mathcal{B},\nu)=\bigotimes_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n)$, while the assignment operation ξ is determined by the following rule

$$\xi(\mathfrak{F})=\begin{cases}\xi_n(\mathfrak{F}), \ \textit{if} \ \ \mathfrak{F}\in\mathrm{St}(\mathcal{B}_n), \ n\in\mathbb{N},\\ \mathrm{prop}(P)\restriction L, \ \ \textit{if}\ \mathfrak{F}=\hat{\mathfrak{F}}, \ \ \hat{\mathfrak{F}}=\textit{Filter}\{-\mathbf{1}_i\,|\,i\in\mathbb{N}\},\end{cases}$$

where $\mathrm{prop}(P)$ is the set of model-theoretic properties associated with the complete theory P.

The idea of the assignment function in the operation $\bigotimes$ for a sequence of semantic types is based on the following algebraic relation for the Boolean algebras

$$\mathrm{St}(\textstyle\bigotimes_{i\in\mathbb{N}}\mathcal{B}_i)=\bigcup_{i\in\mathbb{N}}\mathrm{St}(\mathcal{B}_i)\cup\{\hat{\mathfrak{F}}\},$$

as it is stated in Lemma 2.3(d). Thus, the assignment functions ξ_i cover all ultrafilters defined in the summands. A special value for $\xi(\hat{\mathfrak{F}})$ is required since the functions ξ_i, $i\in\mathbb{N}$, give none value for ξ on the filter $\hat{\mathfrak{F}}$. Furthermore, note that an entry $\bigotimes_{i\in\mathbb{N}}^{[K]}\mathfrak{B}_i$ is possible, where the parameter K represents a set of model-theoretic properties assigned to the particular ultrafilter $\hat{\mathfrak{F}}$.

Give a few statements presenting conditions of decomposability of semantic types and theories in the direct product of a sequence.

First, we give a statement demonstrating a common idea of the decomposition.

Lemma 9.1. [A Decomposition Scheme for Semantic Types] *Let $\mathfrak{B}=(\mathcal{B},\nu,\xi)$ be either a computably enumerable or finitely axiomatizable semantic type, and a_i, $i\in\mathbb{N}$, be an effective sequence of elements in $\mathcal{B}$ satisfying*

$$\text{(a)} \quad a_i\cap a_j=\mathbf{0}, \textit{ for all } i,j\in\mathbb{N} \textit{ such that } i\neq j, \tag{9.2}$$

(b) *there is a unique ultrafilter $\mathfrak{F}$ extending the set $\{-a_i\,|\,i\in\mathbb{N}\}$.*

Then, $\mathfrak{B}\equiv_{AL}\bigotimes_{i\in\mathbb{N}}^{[\xi(\mathfrak{F})]}\mathfrak{B}[a_i]$.

Proof. By immediate checking the definition of the operation of a direct product of a sequence of semantic types. Just elementary properties of Boolean algebras are involved. □

Now, we give a couple of methods to design a decomposition of a theory.

Lemma 9.2. [First Decomposition Scheme for Theories] *Let T be either a computably axiomatizable or finitely axiomatizable theory of signature σ, and Θ_i, $i\in\mathbb{N}$, be an effective sequence of sentences of signature σ such that the following conditions are satisfied:*

$$\text{(a)} \quad T\vdash\Theta_i\to\neg\Theta_j, \textit{ for all } i,j\in\mathbb{N} \textit{ such that } i\neq j, \tag{9.3}$$

(b) *$S=T+\{\neg\Theta_i\,|\,i\in\mathbb{N}\}$ is a complete theory.*

Then, $\pounds(T)\equiv_{AL}\bigotimes_{i\in\mathbb{N}}^{[S]}\pounds(T_i)$, where $T_k=T+\{\Theta_k\}$, $k\in\mathbb{N}$.

Lemma 9.3. [Second Decomposition Scheme for Theories] *Let T be either a computably axiomatizable or finitely axiomatizable theory of signature σ, and Θ_i, $i\in\mathbb{N}$, be an effective sequence of sentences of signature σ satisfying*

$$\text{(a)} \quad T\vdash\Theta_0, \tag{9.4}$$

(b) *$T\vdash\Theta_{k+1}\to\Theta_k$, for all $k\in\mathbb{N}$,*

(c) *$S=T+\{\Theta_i\,|\,i\in\mathbb{N}\}$ is a complete theory.*

Then, $\text{\pounds}(T)\equiv_{AL}\bigotimes^{[S]}_{i\in\mathbb{N}}\text{\pounds}(T_i)$*, where* $T_k=T+\{\Theta_k\,\&\,\neg\Theta_{k+1}\}$*,* $k\in\mathbb{N}$*.*

Proofs for Lemma 9.2 and Lemma 9.3. These statements are simple consequences of some common algebraic properties of Boolean algebras. In essence, decompositions (9.2) and (9.3) are identical; they differ just in notations. Both a_i in (9.2) and Θ_i in (9.3) correspond to an element $(\mathbf{0}_0,\mathbf{0}_1,\dots,\mathbf{0}_{i-1},\mathbf{1}_i,\mathbf{0}_{i+1},\dots)$, while Θ_i in (9.4) correspond to an element $(\mathbf{1}_0,\dots,\mathbf{1}_{i-1},\mathbf{1}_i,\mathbf{0}_{i+1},\dots)$ in the semantic type. In particular, the following dependencies are available between the sentences involved in schemes (9.3) and (9.4):

$$\Theta_k\sim\Theta_k\,\&\,\neg\Theta_{k+1},\quad k\in\mathbb{N},$$

$$\Theta_k\sim\Theta_0\vee\Theta_1\vee\dots\vee\Theta_k,\quad k\in\mathbb{N}.$$

Both Lemma 9.2 and Lemma 9.3 are proved. □

Remark 9.4. A degenerated case $a_i=\mathbf{0}$ is possible in Lemma 9.1 for some indices $i\in\mathbb{N}$; respectively, corresponding members in Lemma 9.2 or Lemma 9.3 may be inconsistent theories. Nevertheless, in any case, statements of Lemma 9.1, Lemma 9.2, and Lemma 9.3 remain to be valid. The direct product will be presented (up to a similarity under *ASL*) by a finite number of applications of a simpler operation of the direct product of two semantic types or theories whenever there are only finitely many non-degenerated members. Of course, the most common is the case when there are infinitely many non-degenerated members or, equivalently, the particular ultrafilter $\mathfrak{F}$ is non-principal. Notice that this remark is applicable to other statements concerning decomposition of either theories or semantic types

Now, we are going to show that the operation of the product of a sequence of semantic types is supported by some natural operations on theories. A common idea of these operations is originated from system of axioms (Ax-0) for theory $T^u_{c.a}$, cf. [9, Sec. 4]. We will specify two different versions of the operation: the first one seems to be more simple, while the second one looks to be maximum common.

First, we describe a *primitive version* of the operation. It concerns the case when a theory of signature

$$\sigma=\{c_0,c_1,\dots,c_k,\dots\}\tag{9.5}$$

defined by the following set of axioms

$$c_i\neq c_j,\quad i,j\in\mathbb{N},\ i\neq j,$$

is taken as a theory for the particular ultrafilter; this theory is denoted by *EQC*.

In the other words, *EQC* is the theory of a countable set of pairwise nonequal constants. Obviously, this theory is computably axiomatizable, categorical in uncountable powers, and does not have finite models. From this we have that the theory *EQC* is complete and decidable.

Let T_n, $n\in\mathbb{N}$, be a sequence of theories, where T_n has a signature σ_n. It is assumed that $\sigma_n\cap\sigma_k=\varnothing$ for all n,k such that $n\neq k$. In general case, if an arbitrary sequence of theories is given intended for the operation, one can rename their signatures making them pairwise disjoint.

Consider the following new signature

$$\sigma'=\{Z^0_k,U^1_k,c_k\,|\,k\in\mathbb{N}\}\cup\sigma_0\cup\sigma_1\cup\dots\cup\sigma_n\cup\dots,$$

where Z_k^0, $k\in\mathbb{N}$, are symbols of nulary predicates. It is assumed that the symbols Z_k, U_k, and c_k for $k\in\mathbb{N}$ do not belong to $\sigma_0\cup\sigma_1\cup\ldots\cup\sigma_n\cup\ldots$.

We construct a theory T' of signature σ' called a *primitive direct product of the sequence* T_n, $n\in\mathbb{N}$, denoted

$$\breve{\bigotimes}_{n\in\mathbb{N}}^{[EQC]} T_n. \tag{9.6}$$

The theory (9.6) is defined by the following set of axioms (Ax-2):

1°. $Z_n\to\neg Z_k,\ \ n,k\in\mathbb{N},\ n\neq k,$

2°. $(\neg Z_0\,\&\ldots\&\,\neg Z_k)\to\bigwedge_{0\leqslant i<j\leqslant k+1}(c_i\neq c_j),\ \ k\in\mathbb{N},$

3°. $Z_n\to(c_n=c_k),\ \ n,k\in\mathbb{N},\ \ n<k,$

4°. $Z_n\to(\forall x)\big[U_n(x)\leftrightarrow(x\neq c_0\,\&\ldots\&\,x\neq c_n)\big],\ n\in\mathbb{N},$

5°. $Z_n\leftrightarrow(\exists x)U_n(x),\ \ n\in\mathbb{N},$

6°. $Z_n\to\big($*all axioms of* T_n *are satisfied in the region* $U(x)\big),\ \ n\in\mathbb{N},$

7°. $Z_n\to($*outside* $U_n(x)$, *all* σ_n*-symbols are defined trivially*$)$, $n\in\mathbb{N},$

8°. $\neg Z_k\to($*all* σ_k*-symbols are defined* c_0*-trivially*$)$, $k\in\mathbb{N}$.

Actually, definition of the operation includes an ambiguity; nevertheless, the target theory (9.6) is defined uniquely up to an isomorphism.

Remark. Later, cf. axioms (Ax-3), we will describe a common version of the operation of a direct product of a sequence of theories. The version we present in (9.6) via (Ax-2) represents a primitive manner to realize a natural idea to link a sequence of theories together. We use an entry $\breve{\bigotimes}_{i\in\mathbb{N}}^{[EQC]}(\ldots)$ for this operation that can be applied to an arbitrary sequence of theories.

Lemma 9.5. *The following assertions hold for the theory* $T'=\breve{\bigotimes}_{n\in\mathbb{N}}^{[EQC]}T_n$ *:*

(a) *for any* $n\in\mathbb{N}$, *theory* $T'+\{Z_n\}$ *is algebraically isomorphic to a constant extension* $T_n\langle c_0,c_1,\ldots,c_n\rangle$ *of theory* T_n,

(b) *theory* $T'\cup\{\neg Z_k\,|\,k\in\mathbb{N}\}$ *is algebraically isomorphic to the theory* EQC.

Proof. By immediate check, based on the system of axioms (Ax-2). □

Now, we approach to the main properties of the operation:

Lemma 9.6. *Given a sequence of theories* T_n, $n\in\mathbb{N}$. *The following statements take place*:

(a) *theory* $T'=\breve{\bigotimes}_{n\in\mathbb{N}}^{[EQC]}T_n$ *is computably axiomatizable* $\Leftrightarrow$ *all theories* T_n, $n\in\mathbb{N}$, *are computably axiomatizable and the sequence* T_n, $n\in\mathbb{N}$, *is computable*,

(b)$\pounds\big(\breve{\bigotimes}_{n\in\mathbb{N}}^{[EQC]}T_n\big)\equiv_{ASL}\bigotimes_{n\in\mathbb{N}}^{[EQC]}\pounds(T_n)$,

Proof. By immediate check, with using Lemma 9.5 and taking into consideration definition of the layer ASL in [10, Def. 1.A]. □

Lemma 9.7. *Given a sequence of semantic types* $\mathfrak{B}_n$, $n\in\mathbb{N}$. *Under any layer* $K\subseteq ASL$, *semantic type* $\mathfrak{B}=\bigotimes_{n\in\mathbb{N}}^{[EQC]}\mathfrak{B}_n$ *is computably axiomatizable* $\Leftrightarrow$ *all types* $\mathfrak{B}_n$, $n\in\mathbb{N}$, *are computably axiomatizable and the sequence* $\mathfrak{B}_n$, $n\in\mathbb{N}$, *is computable.*

Proof. By immediate check, with using Lemma 9.6. □

By applying operation (9.6) to the computable sequence $T^{\star}{}_{\{n\}}$, $n\in\mathbb{N}$, including all possible c.e. theories, cf. [9, Lem. 4.1(b)], we obtain the following theory:

$$T_{c.a.}^{EQC}=\breve{\bigotimes}_{n\in\mathbb{N}}^{[EQC]}T^{\star}{}_{\{n\}}. \tag{9.7}$$

Similarly to $T^{u}_{c.a.}$, cf. [9, (4.1)], theory $T^{EQC}_{c.a.}$ satisfies the universality property relative to the class of all c.a. theories of all possible enumerable signatures. However, complexity of description of theory $T^{u}_{c.a.}$ turns out to be simpler for understanding in comparison with that of theory $T^{EQC}_{c.a.}$.

In order to be prepared to another version of the operation of direct product of a sequence of theories, we have to define a special class of theories intended to present model-theoretic properties of a particular ultrafilter.

Definition 6.E. Let E be a semantic layer (a default value ASL is supposed for E whenever it is not specified explicitly). A theory R of an enumerable signature σ is said to be *inf-dense* under the semantic layer E if there is a computably enumerable set $\Sigma \subseteq SL(\sigma)$ called a *frame* for R that satisfies the following properties:

(a) theory R is complete and decidable,

(b) for any $\Phi \in SL(\sigma)$ satisfying $R \vdash \Phi$, a sentence $\Psi \in SL(\sigma)$ and a computable isomorphism μ can be found satisfying the following properties: $R \vdash \neg\Psi$, $\Sigma \vdash \Psi \to \Phi$, and $₤([\Sigma \cup \{\Psi\}]^{\sigma}) \equiv_E ₤(T^{u}_{c.a.})$ by means of μ; moreover, both a Gödel number of Ψ and an index of the isomorphism μ are found effectively from a Gödel number of the sentence Φ.

Study some common properties of the class of *inf*-dense theories.

Lemma 9.8. *Let R be an inf-dense under E theory of an enumerable signature σ (by definition, this theory is complete and decidable), Σ be a frame of R, and*

$$\Theta_0, \Theta_1, \Theta_2, \ldots, \Theta_n, \ldots \ n \in \mathbb{N}, \tag{9.8}$$

be a Gödel numbering of the set of all sentences of signature σ provable in R. There is a strict monotone computable function $f(x)$, i.e., $(\forall k \in \mathbb{N})[f(k) < f(k+1)]$ holds, such that the following sequence of sentences of signature σ:

$$\Phi_0 = (\exists x)[x = x], \quad \Phi_k = (\Theta_0 \,\&\, \Theta_1 \,\&\ldots\&\, \Theta_{f(k-1)}), \ k \in \mathbb{N} \setminus \{0\},$$

satisfies the following properties:

(a) $\vdash \Phi_0$,

(b) $\vdash \Phi_{k+1} \to \Phi_k$, *for all* $k \in \mathbb{N}$,

(c) $R = [\{\Phi_i \mid i \in \mathbb{N}\}]^{\sigma}$,

(d) *for any k, there is a sentence Ψ_k in $SL(\sigma)$ satisfying $\Sigma + \{\Psi_k\} \vdash \Phi_k \,\&\, \neg\Phi_{k+1}$ together with an isomorphism μ_k presenting the similarity relation $₤(\Sigma + \{\Psi_k\}) \equiv_E ₤(T^{u}_{c.a.})$; moreover, both a Gödel number of Ψ_k and a c.e. index of μ_k are found effectively in k.*

Proof. We use induction by k to define the function $f(x)$ and to build other objects required in Lemma 9.8.

Step $k = 0$ is the basis of induction; at this step we set Φ_0 to be a generally true sentence, $f(x)$ is undefined elsewhere, sentences Ψ_i are not defined.

Now, suppose that k steps have been performed. As a result, sentences $\Phi_0, \ldots, \Phi_k$ and $\Psi_0, \ldots, \Psi_{k-1}$ are defined, and values $f(0), \ldots, f(k-1)$ are chosen; moreover, demands (b) and (c) are satisfied for all cases corresponding to already defined components.

Step k+1. In accordance with formulation of Lemma 9.8, we have to define $f(k)$

and to build Φ_{k+1} and Ψ_k for which the following demands are satisfied:

$$\begin{aligned} &\text{(a)} \quad \vdash \Phi_{k+1} \to \Phi_k, \\ &\text{(b)} \quad \Sigma \cup \{\Psi_k\} \vdash \Phi_k, \\ &\text{(c)} \quad \Sigma \cup \{\Psi_k\} \vdash \neg\Phi_{k+1}, \\ &\text{(d)} \quad \text{₤}(\Sigma + \{\Psi_k\}) \equiv_E \text{₤}(T^u_{c.a.}). \end{aligned} \tag{9.9}$$

We start the construction. First of all, we have $R \vdash \Phi_k$ because all formulas (9.8) are provable from R. Since R is *inf*-dense under E, by applying Definition 6.E to sentence $\Phi = \Phi_k$, we can find a sentence Ψ_k such that:

$$\begin{aligned} &\text{(a)} \quad R \vdash \neg\Psi_k, \\ &\text{(b)} \quad \Sigma \vdash \Psi_k \to \Phi_k, \\ &\text{(c)} \quad \text{₤}(\Sigma + \{\Psi_k\}) \equiv_E \text{₤}(T^u_{c.a.}). \end{aligned} \tag{9.10}$$

We see that (9.10)(b) ensures (9.9)(b), while (9.10)(c) ensures (9.9)(d). Further, $R \vdash \neg\Psi_k$ in (9.10)(a) means that $\{\Theta_0,...,\Theta_i,...\} \vdash \neg\Psi_k$. By Maltsev's Compactness Theorem there is $t \in \mathbb{N}$ such that $\{\Theta_0,...,\Theta_t\} \vdash \neg\Psi_k$. Obviously, we can assume that $t > f(k-1)$. We put $f(k) = t$ and, respectively, we obtain $\Phi_{k+1} = \Theta_0 \& ... \& \Theta_{f(k)}$. In particular, we obtain (9.9)(a). By construction, we have $\Phi_{k+1} \vdash \neg\Psi_k$; therefore, $\Psi_k \vdash \neg\Phi_{k+1}$. Thereby, (9.9)(c) is established. Step $k+1$ is finished. It can be checked that all components of the construction we described (including isomorphism μ_k) are found effectively, i,e, their c.e. indices and Gödel numbers are defined by computable functions from the step parameter k.

Finally, having performed ω steps of the procedure we described, we can state that Part (a) of Lemma 9.8 will also be satisfied because the sequence (9.8) includes all sentences provable in theory R.

Lemma 9.8 is proved. □

Now, having established statement of Lemma 9.8, we turn to the *most common* version of the operation of *direct product* of a sequence of theories.

Let T_n, $n \in \mathbb{N}$, be a computable sequence of theories of enumerable signatures, such that T_n has a signature σ_n. By definition, there is a computable function $e(x)$ such that $T_n = T_{\{e(n)\}}$ for all $n \in \mathbb{N}$. It is assumed that $\sigma_n \cap \sigma_k = \varnothing$ for all n,k. If the condition were failed, we would perform renaming the signatures making them pairwise disjoint.

Consider the following new signature

$$\sigma' = \{Z^0_n,\, U^1, c \,|\, n \in \mathbb{N}\} \cup \tau_0 \cup \tau_1 \cup ... \cup \tau_n \cup ..., \tag{9.11}$$

where Z^0_n, $n \in \mathbb{N}$, are symbols of nulary predicates, while each τ_n is a copy of the signature τ of $T^u_{c.a.}$. Moreover, it is assumed that the parts pointed out in (9.11) are pairwise disjoint.

We construct a theory T' of signature σ' called a *direct product of the sequence* T_n, $n \in \mathbb{N}$, denoted

$$\bigotimes^{[R]}_{n \in \mathbb{N}} T_n. \tag{9.12}$$

We use in the axiomatic a frame set Σ, sentences Φ_i, Ψ_i, and isomorphisms μ_i that were defined in Lemma 9.8. Furthermore, we use propositional variables $\mathcal{Z}_i$ defined in [9, Sec. 4] with theory $T^u_{c.a.}$.

The theory (9.12) is defined by the following set of axioms (Ax-3):

1°. θ, $\theta \in \Sigma$ (Σ is a frame set for the *inf*-dense theory R),

2°. $(\Phi_n \,\&\, \neg\Phi_{n+1}) \to \Psi_n$, $n \in \mathbb{N}$,

3°. $(\Phi_n \,\&\, \neg\Phi_{n+1}) \to \Psi_n \,\&\, \big(\mu_n^{-1}(\mathcal{Z}_{e(n)})\big)$, $n \in \mathbb{N}$.

Notice that, although realization of the introduced operation on a sequence of theories depends on some ambiguity, nevertheless, the target theory T' is defined uniquely up to an algebraic isomorphism.

Based on Lemma 9.8, it is possible to prove the following statement:

Lemma 9.9. *The following properties take place for the theory* $T' = \bigotimes^{[R]}_{n\in\mathbb{N}} T_n$:

(a) *for any* $n \in \mathbb{N}$, *theory* $T' + \{\Phi_n \,\&\, \neg\Phi_{n+1}\}$ *is algebraically isomorphic to a constant extension of* T_n; *particularly, we have* $\text{₤}(T_n) \equiv_K \text{₤}(T' + \{\Phi_n \,\&\, \neg\Phi_{n+1}\})$, *for any semantic layer* $K \subseteq ASL$.

(b) *theory* $T' \cup \{\Phi_k \mid k \in \mathbb{N}\}$ *is algebraically isomorphic to the theory* R.

Proof. An idea of the construction is based on an observation that the sequence of formulas Φ_i, $i \in \mathbb{N}$, provided by Lemma 9.9, together with a frame set Σ for *inf*-dense theory R, satisfy conditions of Lemma 9.3, thereby, ensuring a decomposition of the theory Σ in a direct product of a sequence of theories $T_i = \Sigma \cup \{\Phi_i \,\&\, \neg\Phi_{i+1}\}$, $i \in \mathbb{N}$, with theory R for the particular ultrafilter $\hat{\mathfrak{F}}$. However, the sequence of theories we would obtain in such manner is not suitable (it turns out to be richer: each T_i includes a segment that is equivalent to $T^u_{c.a.}$). To obtain a sequence of theories with the demanded members $T_n = T_{\{e(n)\}}$, $n \in \mathbb{N}$, we have to insert some corrections in both the sequence of formulas and the frame set. Namely, let us define a new corrected sequence of formulas as follows:

$$\Phi'_i = \Phi_{i+1} \vee \Psi_i \,\&\, (\mu^{-1}(\mathcal{Z}_{e(i)})). \tag{9.13}$$

Moreover, we enforce an available frame set Σ up to a theory $T = \Sigma \cup \{\Phi'_0\}$. It is possible to check that, by virtue of Lemma 9.3, we obtain $T = \bigotimes^{[P]}_{n\in\mathbb{N}} T_n$, where $T_i = T + \{\Phi'_i \,\&\, \neg\Phi'_{i+1}\}$, $i \in \mathbb{N}$, and $P = [\{\Phi'_k \mid k \in \mathbb{N}\}]^\sigma = [\{\Phi_k \mid k \in \mathbb{N}\}]^\sigma = R$; thus, Part (b) is satisfied. In accordance with Pard (d) of Lemma 9.8, member Ψ_i in (9.13) makes limitation of the theory up to $T^u_{c.a.}$; moreover, a stronger member $\mu^{-1}(\mathcal{Z}_{e(i)})$ establishes a limitation of the theory up to the demanded theory $T^u_{c.a} + \{\mathcal{Z}_{e(n)}\}$, cf. [9, Lem. 4.1]. In accordance with [9, Lem. 4.1(b,c)], we obtain a computable isomorphism μ between $T_n \rightleftharpoons T_{\{e(n)\}}$ and $T^u_{c.a} + \{\mathcal{Z}_{e(n)}\}$ preserving semantic layer ASL, thus, any its sublayer $K \subseteq ASL$, thereby, establishing all statements demanded in Part (a).

Lemma 9.9 is proved. □

Now, we turn to our main statement:

Lemma 9.10. *Let* R *be an inf-dense theory, and* T_n, $n \in \mathbb{N}$, *be a sequence of theories. The following statements take place*:

(a) *theory* $T' = \bigotimes^{[R]}_{n\in\mathbb{N}} T_n$ *is computably axiomatizable* $\Leftrightarrow$ *all theories* T_n, $n \in \mathbb{N}$, *are computably axiomatizable and the sequence* T_n, $n \in \mathbb{N}$, *is computable*,

(b) $\text{₤}\big(\bigotimes^{[R]}_{n\in\mathbb{N}} T_n\big) \equiv_K \bigotimes^{[R]}_{n\in\mathbb{N}} \text{₤}(T_n)$, *for any semantic layer* $K \subseteq ASL$,

Proof. Immediately, from Lemma 9.9. □

Lemma 9.11. *Let* R *be an inf-dense theory, and* $\mathfrak{B}_n$, $n \in \mathbb{N}$, *be a sequence of semantic types. Under any semantic layer* $K \subseteq ASL$, *the type* $\mathfrak{B} = \bigotimes^{[R]}_{n\in\mathbb{N}} \mathfrak{B}_n$ *is* *computably axiomatizable* $\Leftrightarrow$ *all types* $\mathfrak{B}_n$, $n \in \mathbb{N}$, *are computably axiomatizable and the sequence* $\mathfrak{B}_n$, $n \in \mathbb{N}$, *is computable.*

PROOF. Immediately, from Lemma 9.10. □

A simple properties concerning *inf*-dense theories:

EXERCISE 9.12. *Let a theory* R *of signature* σ *be inf-dense under a semantic layer* K *and* $\Sigma \subseteq SL(\sigma)$ *be an associated set involved in Definition 6.E. Then*

(a) R *is complete and decidable,*

(b) R *is not finitely axiomatizable under* Σ.

HINT. Decidability is followed by Vaught's Theorem from the computable axiomatizability and completeness. Suppose that R is finitely axiomatizable under Σ, and let Φ be a sentence such that $R = \Sigma + \{\Phi\}$. By definition, the theory R is complete. Then, we have $R \vdash \Phi$ but none sentence Ψ is available with the properties stated in the definition because $[\Sigma \cup \{\Phi\}]^\sigma$ determines a complete theory. □

Below, we describe an example of an *inf*-dense theory under any semantic layer $K \subseteq ASL$. This theory is denoted by EQC^* since it is in a close relation with the theory EQC involved in the operation (9.6).

First, we state a simple property of EQC.

EXERCISE 9.13. *Theory* EQC *is not inf-dense.*

HINT. Since signature (9.5) of theory EQC consists of constant symbols only, any theory of this signature is superstable. Thus, the theory EQC cannot satisfy Definition 6.E.

Below, we present an example of an *inf*-dense theory under any semantic layer $K \subseteq ASL$. This theory is denoted by EQC^* since it is in a close relation with the theory EQC involved in the operation (9.6).

Theory EQC^* has the signature

$$\sigma = \{c_0, c_1, \ldots, c_n, \ldots\} \cup \tau_0 \cup \tau_1 \cup \ldots \cup \tau_n \cup \ldots, \tag{9.14}$$

where each τ_n is an enumerable signature which is a copy of the source signature τ of the theory $T^u_{c.a.}$. By $T^u_{c.a.}(\tau_i/\tau)$, we denote an isomorphic copy of $T^u_{c.a.}$ obtained by a rename operation from τ to τ_i. It is assumed that in the signature (9.14) all parts are pairwise disjoint. Axioms of EQC^* include the following sentences:

1°. all axioms of EQC (i.e., $c_i \neq c_j$, for all $i, j \in \mathbb{N}$ such that $i < j$),

2°. each symbol $\mathfrak{s}$ in $\bigcup_{i \in \mathbb{N}} \tau_i$ is defined c_0-trivially.

On this, description of the theory EQC^* is complete.

Lemma 9.14. *The following assertions hold*:

(a) $EQC^* \approx_a EQC$,

(b) EQC^* *is an inf-dense theory under any semantic layer* $K \subseteq ASL$.

PROOF. Part (a) is a trivial consequence of the axioms.

(b) In accordance with Definition 6.E, we have to point out a frame set Σ for R. Define the set Σ as follows:

1°. $(c_k = c_{k+1}) \to (c_{k+1} = c_{k+2})$, $k \in \mathbb{N}$,

2°. $(c_k \neq c_{k+1}) \to (\tau_k\text{-symbols are defined } c_0\text{-trivially})$, $k \in \mathbb{N}$,

3°. $(c_{k-1} = c_k) \to (\tau_k\text{-symbols are defined } c_0\text{-trivially})$, $k \in \mathbb{N}$.

4°. $\Big(\bigwedge_{0 \leqslant i < j \leqslant n} (c_i \neq c_j)\Big) \,\&\, (c_n = c_{n+1}) \to (\Theta)_{U(x)}$, *for all* $\Theta \in SL(\tau_n)$ *s.t.* $\Theta \in$

$T^u_{c.a.}(\tau_i/\tau)$, *where* $U(x) \leftrightharpoons (x \neq c_0 \,\&\ldots\&\, x \neq c_n)$, $n \in \mathbb{N}$,

5°. $\Bigl(\bigwedge_{0\leqslant i<j\leqslant n}(c_i \neq c_j)\Bigr) \,\&\, (c_n = c_{n+1}) \to \tau_n$*-symbols are defined trivially outside of* $U(x)$, *where* $U(x) \leftrightharpoons (x \neq c_0 \,\&\ldots\&\, x \neq c_n)$, $n \in \mathbb{N}$,

In addition, define a set of sentences as follows:

$$\varDelta = \{c_i \neq c_j \mid i,j \in \mathbb{N},\ i<j\}.$$

It is simple to check that $\varSigma \cup \varDelta$ is a system of axioms for EQC^*; thus, EQC^* is computably axiomatizable. In EQC^*, constants c_i, $i \in \mathbb{N}$, represent different elements; moreover, each symbol in each signature τ_n is defined c_0-trivially on the universe set. Thereby, we have $EQC^* \approx_a EQC$. Particularly, EQC^* is complete; thus, it is decidable by Vaught's Theorem.

Now, suppose that $\varPhi \in SL(\sigma)$ and $EQC^* \vdash \varPhi$. By Compactness Theorem, $\varPhi$ is deduced from $\varSigma \cup \varDelta_0$ with an finite set $\varDelta_0 \subseteq \varDelta$. Let just constant symbols $c_0, c_1, \ldots, c_n$ occur in $\varDelta_0$. As $\varPsi$, we choose the following sentence:

$$\varPsi = \Bigl(\bigwedge_{0\leqslant i<j\leqslant n}(c_i \neq c_j)\Bigr) \,\&\, (c_n = c_{n+1}).$$

Furthermore, consider the following formula with one free variable x:

$$U(x) = (x \neq c_0) \,\&\, (x \neq c_1) \,\&\ldots\&\, (x \neq c_n).$$

According to construction, $EQC^* \vdash \neg\varPsi$. Moreover, theory $S = [\varSigma \cup \{\varPsi\}]^\sigma$ represents a constant extension $T\langle c_0, c_1, \ldots, c_n\rangle$ of T in the domain $U(x)$, where T is defined according to Axiom 4°. Therefore, this theory is an isomorphic copy $T^u_{c.a.}(\tau_n/\tau)$ of $T^u_{c.a.}$. As a result, we obtain $\pounds(T^u_{c.a.}) \equiv_K \pounds([\varSigma \cup \{\varPsi\}]^\sigma)$ that is exactly what is required to check Definition 6.E.

Lemma 9.14 is proved. □

Exercise 9.15. *Let a theory* R *of signature* σ *be inf-dense under a semantic layer* K *and* $\varSigma \subseteq SL(\sigma)$ *be an associated set involved in Definition 6.E. Then*

(a) R *is complete and decidable*,

(b) R *is not finitely axiomatizable under* $\varSigma$.

Hint. Decidability is followed by Vaught's Theorem from computable axiomatizability and completeness. Suppose that R is finitely axiomatizable under $\varSigma$, and let $\varPhi$ be a sentence such that $R = \varSigma + \{\varPhi\}$. By definition, the theory R is complete. Then, we would have $R \vdash \varPhi$ but none sentence $\varPsi$ is available with the properties stated in the definition because $[\varSigma \cup \{\varPhi\}]^\sigma$ determines a complete theory. □

Give a few summary statements.

We introduced an operation $\bigotimes^{[P]}_{i\in\mathbb{N}}\mathfrak{B}_i$ of the direct product of a sequence of semantic types $\mathfrak{B}_i$, $i \in \mathbb{N}$, with an extra parameter P pointing out model-theoretic properties assigned to a particular ultrafilter $\hat{\mathfrak{F}}$, cf. (2.3). An advantage of this operation is that it provides a definite result for all input parameters, while its disadvantage is that, in general, it yields an abstract semantic type that might be a type of none theory.

Further, we introduced three operations for sequences of theories: $\breve{\bigotimes}^{[EQ]}_{i\in\mathbb{N}}(\ldots)$ via [9, (Ax-0)], $\breve{\bigotimes}^{[EQC]}_{i\in\mathbb{N}}(\ldots)$ via (Ax-2), and $\bigotimes^{[R]}_{i\in\mathbb{N}}(\ldots)$ via (Ax-3); in the latter case, an extra parameter R can be an arbitrary *inf*-dense theory. Moreover, effective versions of these operations are meant by default (they are normally applied to just computable sequences of theories; although, abstract versions without effectiveness can be considered, whenever the reader wish to do without computability). Version of the operation on a sequence of theories defined via the axioms (Ax-2) represents (in some sense) a particular case of a more general version of the operation defined via the axioms (Ax-3); namely, based on Lemma 9.14 together with some internal details of theories defined via (Ax-2) and (Ax-3), we can establish the following important relation between the two versions of the operation:

$$\breve{\bigotimes}^{[EQC]}_{i\in\mathbb{N}} T_i \equiv_{ASL} \bigotimes^{[EQC^*]}_{i\in\mathbb{N}} T_i. \qquad (9.15)$$

As for the operation defined via (Ax-0) in [9], it plays the role of a prototype of idea for further constructions. Main disadvantage of (Ax-0) is that countably many particular ultrafilters appear in this operation instead of a single ultrafilter in further versions of the operation. Key idea of the operation (Ax-2) as well as further ones is to ensure that a single particular ultrafilter would appear in the operation.

10 Interdependence between the operations

Give an analog of Lemma 3.6 for the semantic types.

Lemma 10.1. *Let $L\subseteq ASL$ be a semantic layer and P be an arbitrary complete theory of an enumerable signature. Then, for any semantic types under the semantic layer L the following relations are satisfied*:

(a) $(\mathcal{B}_1,\nu_1,\xi_1)\otimes(\mathcal{B}_2,\nu_2,\xi_2)\equiv_L(\mathcal{B}_2,\nu_2,\xi_2)\otimes(\mathcal{B}_1,\nu_1,\xi_1)$,

(b) $\Big((\mathcal{B}_1,\nu_1,\xi_1)\otimes(\mathcal{B}_2,\nu_2,\xi_2)\Big)\otimes(\mathcal{B}_3,\nu_3,\xi_3)\equiv_L$
$(\mathcal{B}_1,\nu_1,\xi_1)\otimes\Big((\mathcal{B}_2,\nu_2,\xi_2)\otimes(\mathcal{B}_3,\nu_3,\xi_3)\Big)$,

(c) $\bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n,\xi_n)\equiv_L\bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_{f(n)},\nu_{f(n)},\xi_{f(n)})$,
for any computable permutation f of the set $\mathbb{N}$,

(d) $\bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_n,\nu_n,\xi_n)\equiv_L(\mathcal{B}_0,\nu_0,\xi_0)\otimes\bigotimes^{[P]}_{n\in\mathbb{N}}(\mathcal{B}_{n+1},\nu_{n+1},\xi_{n+1})$,

(e) $(\mathcal{B},\nu,\xi)\equiv_L(\mathcal{B},\nu,\xi)[a]\otimes(\mathcal{B},\nu,\xi)[-a]$, *for any* $a\in\mathcal{B}$.

PROOF. Immediately, based on corresponding parts of Lemma 3.6. □

Conclusion

The works [3], [4], and [5] represent a conceptual framework of the finitary first-order combinatorics. Main aim of this approach is to find characterization of the structure of the Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures. In this paper, we introduce the concept of a semantic type and consider main properties of such types within the context of the first-order finitary and infinitary combinatorics. Natural operations on theories and similar operations on semantic types are defined and studied. An important point is that the class of abstract semantic types is wider in comparison with that in the class of types realized in theories. Moreover, the operation of the direct product of a sequence has a more regular definition on semantic types in comparison with the case of theories. This opens up possibility of constructing semantic types with the universality conditions and exploring their properties. Later, having found special criteria of realizability of the resulting semantic types in theories, we manage to obtain appropriate description of the class of theories with the universality condition.

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