Characterization of universal semantic types of computably axiomatizable theories in the framework of the first-order combinatorial approach

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Abstract

In the work, we study the semantic types representing the structure of the generalized Tarski-Lindenbaum algebras of first-order theories. We investigate methods of constructing of universal semantic types and investigate their properties. We describe universal finitely axiomatizable and computably axiomatizable semantic types that give a characterization of the generalized Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures under finitary and infinitary semantic layers .

Keywords: *First-order logic; Tarski-Lindenbaum algebra; model theoretic property; semantic type of the theory; computable isomorphism.*

Results of the works [1] and [2] describe some special methods of constructing computable isomorphisms between the Tarski-Lindenbaum algebras of predicate calculi $PC(\sigma_1)$ and $PC(\sigma_2)$ of finite rich signatures σ_1 and σ_2. Finite-to-finite signature reduction procedure is involved in the main construction of the work [2]; it would be natural to call these transformations of theories as methods of *finitary first-order combinatorics*. On the other hand, an available version of the universal construction of finitely axiomatizable theories is involved in the proof of the main statement of the work [1]; thus, it would be natural to call these transformations of theories as methods of *infinitary first-order combinatorics*. It is important that the pointed out classes of transformations of theories preserve definite semantic layers of model-theoretic properties. Methods of finitary first-order combinatorics preserve a semantic layer called *finitary*, while methods of infinitary first-order combinatorics preserve a semantic layer called *infinitary*.

The works [3], [4], and [5] represent a conceptual framework of finitary first-order combinatorics. The main goal of this approach is characterization of the structure of the Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures. In this work, we introduce natural classes of semantic types and study methods of constructing of universal semantic types. In comparison with the works [1] and [2], an advanced characterization of isomorphism types of universal semantic types is found. The obtained results give the possibility to obtain complete characterization of the generalized Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures under finitary and infinitary semantic layers of model-theoretic properties.

Preliminaries

We consider theories in *first-order predicate logic* with *equality* and use general concepts of model theory, algorithm theory, constructive models and Boolean algebras that can be found in [6], [7], and [8]. Main technical concept used in this work are found in [9] and [10], while definitions concerning semantic layers within the first-order combinatorial approach are found in [11]. Generally, *incomplete* theories are considered. In this work, we consider just signatures admitting Gödel's numberings of formulas. Such a signature is called *enumerable*.

A finite signature is called *rich* if it contains at least one n-ary predicate or function symbol for $n\geqslant 2$, or two symbols of unary functions. In description of a signature, capital letters are used for predicate symbols, while lowercase letters for function and constant symbols. Moreover, superscripts mean arities of corresponding symbols. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_k(\sigma)$ is the set of all formulas of signature σ with free variables $x_0,\ldots,x_{k-1}$, $SL(\sigma)$ is the set of all sentences of signature σ. By $L(T)$, we denote the Boolean algebra of sentences of theory T modulo the equivalence relation in T. It is called the *Tarski-Lindenbaum algebra* of T. By $\mathcal{L}(T)$, we denote the Tarski-Lindenbaum algebra $L(T)$ considered together with a Gödel numbering γ of the set of sentences; thereby, the concept of a computable isomorphism is applicable to such objects. By *GR*, we denote the Graph theory of signature $\sigma_{_{GR}}=\{\Gamma^2\}$ defined by axioms $(\forall x)\neg\Gamma(x,x)$ and $(\forall x)(\forall y)[\Gamma(x,y)\leftrightarrow\Gamma(y,x)]$, while *GRE* denotes an extension of *GR* defined by extra axioms $(\exists x,y)\Gamma(x,y)$ and $(\exists x,y)\big[(x\neq y)\,\&\,\neg\Gamma(x,y)\big]$. For a set of natural numbers, c.e. means *computably enumerable*. For a theory, c.a. means *computably axiomatizable*, while f.a. means *finitely axiomatizable*.

Using Post's numbering W_n, $n\in\mathbb{N}$, for the family of all computably enumerable sets, we organize an effective numbering for the class of all computably axiomatizable theories. Two versions of indices are possible. The first one presents c.e. indices of c.a. theories of an enumerable or finite signature σ. If a theory T of signature σ is defined by set of axioms $\{\Phi_i\,|\,i\in W_m\}$, the number m is called a *computably enumerable index* or simply *c.e. index* of T. The second version represents weak indices for theories of different enumerable signatures $\sigma\subseteq\sigma^\infty$. For a given $m\in\mathbb{N}$, we consider a set of axioms $\Sigma=\{\Phi_i^\infty\,|\,i\in W_m\}$ and construct theory $T=[\Sigma]^\star$. The number m is called a *weak computably enumerable index* or simply *weak c.e. index* of the theory T. As for finitely axiomatizable theories, any such theory F is defined by a finite system A of axioms and therefore, by a single formula Φ which is a conjunction of formulas from A. If a f.a. theory F of signature σ is defined by an axiom Φ_m, the number m is called Gödel's number or simply *strong index* of F. For a given $m\in\mathbb{N}$, we consider an f.a. theory $F=[\Phi_m]^\star$. This number m is called a *universal Gödel's number* or simply *universal strong index* of the theory F. By $T^\sigma{}_{\{n\}}$ we denote a theory of signature σ with c.e. index n, while $T^\star{}_{\{n\}}$ is a theory with weak c.e. index n. Furthermore, by $F^\sigma{}_{\{n\}}$ we denote a f.a. theory of signature σ with Gödel's number n, while $F^\star{}_{\{n\}}$ is an f.a. theory with weak strong index n.

1 Introduction in the concept of a universal semantic type

In this section, universal semantic types are studied, which are used in a characterization of the Tarski-Lindenbaum algebra of first-order Predicate Calculi. Generally, finitely axiomatizable semantic types under the finitary semantic layer *ACL* and computably axiomatizable types under the infinitary semantic layer *MQL* $\subseteq$ *MQL* are considered, while some of the results concern to other semantic layers.

We introduce a block of principal definitions.

Definition 1.A. Given a layer L of model-theoretic properties (a default value *ACL* is supposed for L whenever it is not specified explicitly in an application). A type $\mathfrak{B}=(\mathcal{B},\nu,\xi)$ in *SemTypF*(L) is called *weak* $\mathcal{F}$*-universal* under the semantic layer L if any $\mathcal{F}$-type $\mathfrak{B}'$ under L can be presented as $\mathfrak{B}[a]$ for some $a\in|\mathfrak{B}|$. A type $\mathfrak{B}=(\mathcal{B},\nu,\xi)$ in *SemTypF*(L) is called $\mathcal{F}$*-universal* under a semantic layer L, if, effectively in an index of any $\mathcal{F}$-type $\mathfrak{B}'$ under L, the type $\mathfrak{B}'$ can be presented as $\mathfrak{B}[a]$ for some $a\in|\mathfrak{B}|$; more precisely, there are general computable functions $g(n)$ and $h(n,t)$ satisfying the following properties:

$$\mathcal{F}^\star{}_{\{n\}} \equiv_L \mathfrak{B}[\nu(g(n))],\ \textit{for all } n\in\mathbb{N};\ \textit{moreover, the function } (\lambda t)h(n,t) \textit{ represents this computable isomorphism.} \tag{1.1}$$

Definition 1.B. Given a layer M of model-theoretic properties (a default value *MQL* is supposed for M whenever it is not specified explicitly in an application). A type $\mathfrak{B}=(\mathcal{B},\nu,\xi)$ in *SemTypE*(M) is called *weak* $\mathcal{E}$*-universal* under the semantic layer M if any $\mathcal{E}$-type $\mathfrak{B}'$ under M can be presented as $\mathfrak{B}[a]$ for some $a\in|\mathfrak{B}|$. A type $\mathfrak{B}=(\mathcal{B},\nu,\xi)$ in *SemTypE*(M) is called $\mathcal{E}$*-universal* under a semantic layer M, if, effectively in an index of any $\mathcal{E}$-type $\mathfrak{B}'$ under M, the type $\mathfrak{B}'$ can be presented as $\mathfrak{B}[a]$ for some $a\in|\mathfrak{B}|$; more precisely, there are general computable functions $g(n)$ and $h(n,t)$ satisfying the following properties:

$$\mathcal{E}^\star{}_{\{n\}} \equiv_M \mathfrak{B}[\nu(g(n))],\ \textit{for all } n\in\mathbb{N};\ \textit{moreover, the function } (\lambda t)h(n,t) \textit{ represents this computable isomorphism.} \tag{1.2}$$

Lemma 1.1. *The following assertions hold for any finite rich signature* σ:

(a) *semantic type* $\pounds(PC(\sigma))$ *is weak* $\mathcal{F}$*-universal under the semantic layer ACL (thus, under any its sublayer* $L\subseteq$ *ACL*),

(b) *semantic type* $\pounds(PC(\sigma))$ *is weak* $\mathcal{E}$*-universal under the semantic layer MQL (thus, under any its sublayer* $K\subseteq$ *MQL*).

Proof. (a) Obviously, $\pounds(PC(\sigma))$ is a $\mathcal{F}$-type under *ACL*. By [9, Th. 1.1], there is a sentence a such that $\pounds(PC(\sigma))[a]$ is equivalent to *GRE*. By [9, Lem. 0.2], we obtain exactly what is required.

(b) By [9, Lem. 6.1] we obtain that $\mathcal{E}$-type $\pounds(T^u_{c.a.})$ is an $\mathcal{F}$-type under *MQL*. By Definition 1.B, there is a sentence a such $\pounds(PC(\sigma))[a]$ is equivalent to $T^u_{c.a.}$ under *MQL*. By virtue of [9, Lem. 4.1], theory $T^u_{c.a.}$ has the universality property under the class of all $\mathcal{E}$-types; from this, we can easily obtain exactly what is required.

Lemma 1.1 is proved. □

Lemma 1.2. *The following assertions hold*:

(a) *For any semantic layer* $L \subseteq ACL$ *and any type* $\mathfrak{B} \in SemTypF(L)$ *we have:* $\mathfrak{B}$ *is weak* $\mathcal{F}$*-universal under* $L \Leftrightarrow \mathfrak{B}$ *is* $\mathcal{F}$*-universal under* L.

(b) *For any semantic layer* $K \subseteq ASL$ *and any type* $\mathfrak{B} \in SemTypE(K)$ *we have:* $\mathfrak{B}$ *is weak* $\mathcal{E}$*-universal under* $K \Leftrightarrow \mathfrak{B}$ *is* $\mathcal{E}$*-universal under* K.

Proof. (a) The back implication is obvious. Furthermore, If $\mathfrak{B}$ is weak $\mathcal{F}$-universal under L, there is an element $a \in |\mathfrak{B}|$ such that $\mathfrak{B}[a] \equiv_L \text{₤}(GRE)$. Then, the procedure RedLev described in [9, Lem. 1.3] provides the effectiveness property required for the universality of $\mathfrak{B}$. Part (b) is proved by the same scheme as in [9, Lem. 4.1] with theory $T^u_{c.a.}$ instead of *GRE*. □.

Lemma 1.3. *The following assertions hold*:

(a) *Let a semantic* $\mathcal{F}$*-type* $\mathfrak{B}$ *be* $\mathcal{F}$*-universal under a semantic layer* N*. Then,* $\mathfrak{B}$ *is* $\mathcal{F}$*-universal under any smaller semantic layer* $L \subseteq N$.

(b) *Let a semantic* $\mathcal{E}$*-type* $\mathfrak{B}$ *be* $\mathcal{E}$*-universal under a semantic layer* M*. Then,* $\mathfrak{B}$ *is* $\mathcal{E}$*-universal under any smaller semantic layer* $K \subseteq M$.

Proof. Immediately, from definition. □.

Lemma 1.4. *Semantic type* ₤(GRE) *of graph theory is* $\mathcal{F}$*-universal under the semantic layer* ACL *(thus, under any its sublayer* $L \subseteq ACL$*).*

Proof. Consider the semantic type $\mathfrak{B} = \text{₤}(GRE)$ under the semantic layer *ACL*. This is a finitely axiomatizable type which is universal by [9, Lem. 1.3]. As a matter of fact, just transformations of a fixed level $e=2$ are sufficient to support the required properties (an alternative method is to use a finite-to-finite signature reduction procedure, cf. [9, Th. 1.1]. □.

2 Rich computable sequences of semantic types

Introduce the concept of computability for a family of semantic types.

Definition 2.A. A sequence $\mathfrak{B}_n$, $n \in \mathbb{N}$, of $\mathcal{F}$-types under a semantic layer L is called *computable* if there is a general computable function $f(x)$ such that $\mathfrak{B}_n \equiv_L \mathcal{F}^{\star}{}_{\{f(n)\}}$ for all $n \in \mathbb{N}$.

Definition 2.B. A sequence of $\mathcal{E}$-types $\mathfrak{B}_n$, $n \in \mathbb{N}$, under a semantic layer L is called *computable*, if there is a general computable function $f(x)$ such that $\mathfrak{B}_n \equiv_L \mathcal{E}^{\star}{}_{\{f(n)\}}$ for all $n \in \mathbb{N}$.

Введём несколько классов вычислимых богатых последовательностей семантических типов.

Definition 2.C. A sequence of $\mathcal{F}$-types $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$, is called $\mathcal{F}$-*rich* under a semantic layer L, if it is computable and, effectively in an index, any $\mathcal{F}$-type $\mathfrak{B}$ under L can be presented as $\mathfrak{B}_i[a]$ for some $a \in |\mathfrak{B}_i|$ with $i \geqslant k$ for arbitrarily large k; more precisely, if there are general computable functions $f(n,x)$, $g(n,x)$ and $S(n,x,t)$ satisfying the following properties:

$$f(n,x) \geqslant x, \quad \textit{for all } n,x \in \mathbb{N}, \tag{2.1}$$

$\mathcal{F}^{\star}{}_{\{n\}} \equiv_L \mathfrak{B}_{f(n,x)}[\nu_{f(n,x)}(g(n,x))]$, *for all* $n,x \in \mathbb{N}$; *moreover, the function* $(\lambda t)h(n,x,t)$ *represents this computable isomorphism.*

Definition 2.D. A sequence of $\mathcal{E}$-types $\mathfrak{B}_n = (\mathcal{B}_n, \nu_n, \xi_n)$, $n \in \mathbb{N}$, is called $\mathcal{E}$-*rich* under a semantic layer M, if it is computable and, effectively in an index, any $\mathcal{E}$-type $\mathfrak{B}$ under M can be presented as $\mathfrak{B}_i[a]$ for some $a\in|\mathfrak{B}_i|$ with $i\geqslant k$ for arbitrarily large k; more precisely, if there are general computable functions $f(n,x)$, $g(n,x)$ and $S(n,x,t)$ satisfying the following properties:

$$f(n,x)\geqslant x, \quad \textit{for all } n,x\in\mathbb{N}, \tag{2.2}$$

$\mathcal{E}^\star{}_{\{n\}}\equiv_M \mathfrak{B}_{f(n,x)}[\nu_{f(n,x)}(g(n,x))]$, *for all* $n,x\in\mathbb{N}$; *moreover, the function* $(\lambda t)h(n,x,t)$ *represents this computable isomorphism.*

First, introduce a technical definition.

Definition 2.E. A sequence $\mathfrak{B}_n=(\mathcal{B}_n,\nu_n,\xi_n)$, $n\in\mathbb{N}$, of $\mathcal{F}$-types under a semantic layer L is called *uniformly $\mathcal{F}$-universal* under L, if it is computable and the universality condition is satisfied uniformly; more precisely, if there are general computable functions $g(m,n)$ and $S(m,n,t)$ satisfying the following properties:

$$\mathcal{F}^\star{}_{\{n\}}\equiv_L \mathfrak{B}_m[\nu(g(m,n))], \textit{ for all } m,n\in\mathbb{N}; \textit{ moreover, the function} \tag{2.3}$$

$(\lambda t)h(m,n,t)$ *represents this computable isomorphism.*

Definition 2.F. A sequence $\mathfrak{B}_n=(\mathcal{B}_n,\nu_n,\xi_n)$, $n\in\mathbb{N}$, of $\mathcal{E}$-types under a semantic layer M is called *uniformly $\mathcal{E}$-universal* under the semantic layer M, if it is computable and the universality condition is satisfied uniformly; more precisely, if there are general computable functions $g(m,n)$ and $S(m,n,t)$ satisfying the following properties:

$$\mathcal{E}^\star{}_{\{n\}}\equiv_M \mathfrak{B}_m[\nu(g(m,n))], \textit{ for all } m,n\in\mathbb{N}; \textit{ moreover, the function} \tag{2.4}$$

$(\lambda t)h(m,n,t)$ *represents this computable isomorphism.*

Give some natural concepts of representability of sequences.

Definition 2.G. A computable sequence of $\mathcal{F}$-types $\mathfrak{B}_n$, $n\in\mathbb{N}$, is called *$\mathcal{F}$-representative* under a semantic layer L if the following conditions are satisfied: (a) the sequence has effective cylindric properties, i.e., there are computable functions $f(n,k)$ and $p(n,k,x)$ satisfying $f(n,0)=n$, $f(n,k+1)>f(n,k)$, and $\mathcal{F}^\star{}_{\{n\}}\equiv_M \mathfrak{B}_{f(n,k)}$ for all $n,k\in\mathbb{N}$, moreover, the function $\lambda x p(n,k,x)$ represents this computable isomorphism; (b) the sequence presents effectively all possible $\mathcal{F}$-types under L, i.e., there are computable functions $h(n)$ and $q(n,x)$ such that $\mathfrak{B}_n\equiv_L \mathcal{F}^\star{}_{\{h(n)\}}$ for all $n\in\mathbb{N}$, moreover, the function $\lambda x q(n,x)$ represents this computable isomorphism.

Definition 2.H. A computable sequence of $\mathcal{E}$-types $\mathfrak{B}_n$, $n\in\mathbb{N}$, is called *$\mathcal{E}$-representative* under a semantic layer K if the following conditions are satisfied: (a) the sequence has effective cylindric properties, i.e., there are computable functions $f(n,k)$ and $p(n,k,x)$ satisfying $f(n,0)=n$, $f(n,k+1)>f(n,k)$, and $\mathcal{E}^\star{}_{\{n\}}\equiv_K \mathfrak{B}_{f(n,k)}$ for all $n,k\in\mathbb{N}$, moreover, the function $\lambda x p(n,k,x)$ represents this computable isomorphism; (b) the sequence presents effectively all possible $\mathcal{E}$-types under K, i.e., there are computable functions $h(n)$ and $q(n,x)$ such that $\mathfrak{B}_n\equiv_K \mathcal{E}^\star{}_{\{h(n)\}}$ for all $n\in\mathbb{N}$, moreover, the function $\lambda x q(n,x)$ represents this computable isomorphism.

Establish computability of the set of all finitely axiomatizable types.

Theorem 2.1. *The following assertions hold*:

(a) *The set of all finitely axiomatizable semantic types under an arbitrary semantic layer L, which are types of finitely axiomatizable theories of a fixed finite signature σ, is computable.*

(b) *The set of all finitely axiomatizable semantic types under an arbitrary semantic layer L, which are types of all possible finitely axiomatizable theories of any finite signatures, is computable.*

Proof. Consider the following sequences of semantic types:

$$\text{(a)}\quad \mathcal{F}^{\star}{}_{\{n\}} = Ł(F^{\star}{}_{\{n\}}),\ n\in\mathbb{N}, \tag{2.5}$$

$$\text{(b)}\quad \mathcal{F}^{\sigma}{}_{\{n\}} = Ł(F^{\sigma}{}_{\{n\}}),\ n\in\mathbb{N}.$$

By applying [9, Lem. 3.1], we obtain exactly what is required. □

Similar statement concerning computably axiomatizable types.

Theorem 2.2. *The set of all computably axiomatizable semantic types under an arbitrary semantic layer K, which are types of all possible computably axiomatizable theories of any enumerable signatures, is computable.*

Proof. Consider the following sequences of semantic types:

$$\mathcal{E}^{\star}{}_{\{n\}} = Ł(T^{\star}{}_{\{n\}}),\ n\in\mathbb{N}. \tag{2.6}$$

By applying [9, Lem. 3.2], we obtain exactly what is required. □

Theorem 2.3. *Given semantic layers L and K such that $L\subseteq ACL$ and $K\subseteq MSL$. The following assertions take place*:

(a) *there is a computable sequence of $\mathcal{F}$-types that is $\mathcal{F}$-representative under the layer L; in particular, both sequences* (2.5)*(a) and* (2.5)*(b) has these properties*;

(b) *any two computable $\mathcal{F}$-representative sequences $\mathcal{F}_i$, $i\in\mathbb{N}$, and $\mathcal{F}'_i$, $i\in\mathbb{N}$, of $\mathcal{F}$-types under the layer L are equivalent to each other; more precisely, there are computable functions $p(n)$ and $f(n,x)$, such that p is a permutation of the set $\mathbb{N}$ satisfying $\mathcal{F}_n \equiv_L \mathcal{F}'_{p(n)}$ for all $n\in\mathbb{N}$; moreover, the function $\lambda x f(n,x)$ represent an isomorphism for the pointed out similarity relation.*

(c) *there is a computable sequence of $\mathcal{E}$-types that is $\mathcal{E}$-representative under the layer K; in particular, the sequence* (2.6) *has these properties*;

(d) *any two computable $\mathcal{E}$-representative sequences $\mathcal{E}_i$, $i\in\mathbb{N}$, and $\mathcal{E}'_i$, $i\in\mathbb{N}$, of $\mathcal{E}$-types under the layer K are equivalent to each other; more precisely, there are computable functions $p(n)$ and $f(n,x)$, such that p is a permutation of the set $\mathbb{N}$ satisfying $\mathcal{E}_n \equiv_K \mathcal{E}'_{p(n)}$ for all $n\in\mathbb{N}$; moreover, the function $\lambda x f(n,x)$ represent an isomorphism for the pointed out similarity relation.*

Proof. Part (a) is proved by immediate checking Definition 2.G for semantic types of sequences of theories [9, (3.1)(a)], while for [9, (3.1)(b)], we have to use additionally a finite-to-finite signature reduction procedure, cf. [9, Th. 1.2]. Part (b) is established by applying method of proof of Myhill's Theorem in algorithm theory, cf. [7, Sec. 7.4].

Part (c) is proved by immediate checking Definition 2.H for semantic types of the sequence of theories [9, (3.1)(d)]. Part (d) is established by applying method of proof of the Myhill Theorem, [7, Sec. 7.4]. □

The following statement represents an effective version of [10, Lem. 4.2].

Theorem 2.4. [$\mathbb{U}$] *Given semantic layers L and K such that $L\subseteq ACL$ and $K\subseteq MSL$. Let $\mathcal{F}_i$, $i\in\mathbb{N}$, be a computable $\mathcal{F}$-representative sequence of $\mathcal{F}$-types* *under L and $\mathcal{E}_i$, $i\in\mathbb{N}$, a computable $\mathcal{E}$-representative sequence of $\mathcal{E}$-types under the layer K. There is a computable permutation $p:\mathbb{N}\to\mathbb{N}$ together with a computable function $h(n,x)$ such that $\mathcal{F}_i\equiv_{L\cap K\cap MQL}\mathcal{E}_{p(i)}$ for all $n\in\mathbb{N}$; moreover, the function $\lambda x h(n,x)$ represents an isomorphism for the pointed out similarity relation.*

Proof. By applying an available effective version of the universal construction, cf. [9, St. 2.2], we can build an effective embedding of sequence $\mathcal{F}_i$, $i\in\mathbb{N}$, into sequence $\mathcal{E}_{\{k\}}$, $k\in\mathbb{N}$; by Theorem 2.3, the former is isomorphic to the pointed out sequence $\mathcal{E}_i$, $i\in\mathbb{N}$. A back embedding can be established by a similar method with using [9, Lem. 3.5(a′)]. Finally, based on the cylindric properties of the sequences $\mathcal{F}_i$, $i\in\mathbb{N}$, and $\mathcal{E}_i$, $i\in\mathbb{N}$, stated in Theorem 2.3, we can build the demanded computable bijection by applying method of proof of the Myhill Theorem, [7, Sec. 7.4].

Theorem 2.4 is proved. □

Exercise 2.5. A computable sequence $\mathfrak{B}_i$, $i\in\mathbb{N}$, of finitely axiomatizable semantic types under $L\subseteq ACL$ can represent c.e. indices of $\mathcal{F}$-types if and only if it is $\mathcal{F}$-representative under L. Similarly, a computable sequence $\mathfrak{B}_i$, $i\in\mathbb{N}$, of computably axiomatizable types under $K\subseteq ASL$ can represent c.e. indices of $\mathcal{E}$-types if and only if it is $\mathcal{E}$-representative under K.

Hint. Use Theorem 2.3 together with definitions of indices for $\mathcal{F}$-types and $\mathcal{E}$-types, cf. [9, Sec. 6].

3 Isomorphism criteria for semantic types

The following facts take place.

Lemma 3.1. *Let L be a semantic layer, P be a complete theory, and the following conditions hold:*

(a) $\mathfrak{B}_i$, $i\in\mathbb{N}$, *is a uniformly $\mathcal{F}$-universal sequence of $\mathcal{F}$-types under L,*

(b) $\mathfrak{B}'_j$, $j\in\mathbb{N}$, *is a uniformly $\mathcal{F}$-universal sequence of $\mathcal{F}$-types under L,*

(c) $\mathfrak{B}^*$ *is an arbitrary $\mathcal{F}$-type under the layer L.*

Then, the isomorphism $\bigotimes^{[P]}_{i\in\mathbb{N}}\mathfrak{B}_i\equiv_L\mathfrak{B}^*\otimes\bigotimes^{[P]}_{j\in\mathbb{N}}\mathfrak{B}'_j$ takes place.

A similar statement for computably axiomatizable types:

Lemma 3.2. *Let M be a semantic layer, P be a complete theory, and the following conditions hold:*

(a) $\mathfrak{B}_i$, $i\in\mathbb{N}$, *is a uniformly $\mathcal{E}$-universal sequence of $\mathcal{E}$-types under M,*

(b) $\mathfrak{B}'_j$, $j\in\mathbb{N}$, *is a uniformly $\mathcal{E}$-universal sequence of $\mathcal{E}$-types under M,*

(c) $\mathfrak{B}^*$ *is an arbitrary $\mathcal{E}$-type under the layer M.*

Then, the isomorphism $\bigotimes^{[P]}_{i\in\mathbb{N}}\mathfrak{B}_i\equiv_M\mathfrak{B}^*\otimes\bigotimes^{[P]}_{j\in\mathbb{N}}\mathfrak{B}'_j$ takes place.

Proof to Lemma 3.1. For the sake of unification, the semantic type $\mathfrak{B}^*$ is denoted by $\mathfrak{B}'_{-1}$. So, we have two sequences of $\mathcal{F}$-types

$$\mathfrak{B}_i=(\mathcal{B}_i,\nu_i,\xi_i),\ i\in\mathbb{N},\quad \mathfrak{B}'_i=(\mathcal{B}'_i,\nu'_i,\xi'_i),\ i\in\mathbb{N}\cup\{-1\},$$

satisfying the conditions of Lemma 3.1. Our goal is to construct a computable isomorphism between the following semantic types

$$\mathfrak{B}=\bigotimes\nolimits^{[P]}_{i\geqslant 0}(\mathcal{B}_i,\nu_i,\xi_i),\quad \mathfrak{B}'=\bigotimes\nolimits^{[P]}_{i\geqslant -1}(\mathcal{B}'_i,\nu'_i,\xi'_i). \tag{3.3}$$

For this purpose, a step-by-step procedure is described that determines a sequence of partial isomorphisms λ_i and μ_i for all $i\in\mathbb{N}$. Scheme in Fig. 3.1 represents main details of the construction. The construction is effective. That is, the numbers of elements involved in the process are found effectively; moreover, all partial isomorphisms λ_i and μ_i are supposed to be computable and their indices are found effectively.

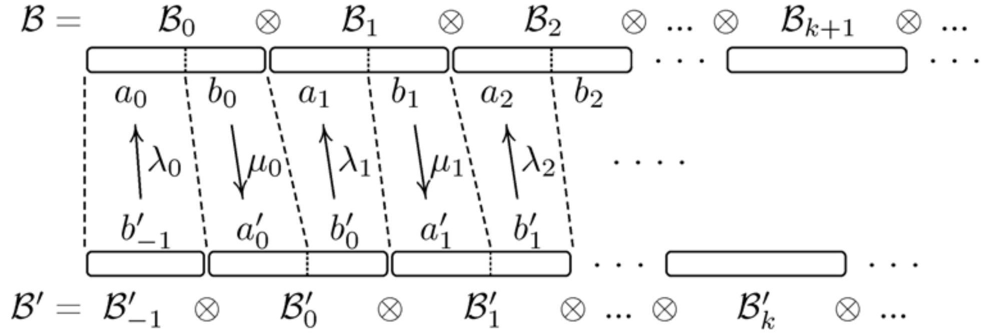


Fig. 3.1. Scheme presenting an isomorphism constructing method

Denote by $\mathbf{1}_i$ an element of the direct product $\bigotimes_{i\geqslant 0}\mathcal{B}_i$ corresponding to $\mathbf{1}$ in its member $\mathcal{B}_i$. Furthermore, denote by $\mathbf{1}'_i$ an element of the direct product $\bigotimes_{i\geqslant -1}\mathcal{B}'_i$ corresponding to $\mathbf{1}$ in its member $\mathcal{B}'_i$.

Description of the construction.

Step $t=-1$. Denote $b'_{-1}=\mathbf{1}'_{-1}$. Basing on the fact that the type $\mathfrak{B}_0$ satisfies the universality property, find an element $a_0\subseteq\mathbf{1}_0$ for which there is an isomorphism $\lambda_0:\mathfrak{B}'[b'_{-1}]\to\mathfrak{B}[a_0]$ under the semantic layer L.

To the beginning of the step t, let all previous steps $-1,0,\ldots,t-1$ be already performed, and as a result, an isomorphism $\lambda_t:\mathfrak{B}[b'_{t-1}]\to\mathfrak{B}[a_t]$ under the semantic layer L is already defined.

Step $t\geqslant 0$. It consists of two stages. At the first stage: denote $b_t=\mathbf{1}_t\smallsetminus a_t$; basing on the fact that the type $\mathfrak{B}'_t$ satisfies the universality property, find an element $a'_t\subseteq\mathbf{1}'_t$ for which there is an isomorphism $\mu_t:\mathfrak{B}[b_t]\to\mathfrak{B}'[a'_t]$ under the semantic layer L. At the second stage: denote $b'_t=\mathbf{1}'_t\smallsetminus a'_t$; basing on the fact that the type $\mathfrak{B}_{t+1}$ satisfies the universality property, find an element $a_{t+1}\subseteq\mathbf{1}_{t+1}$ for which there is an isomorphism $\lambda_{t+1}:\mathfrak{B}'[b'_t]\to\mathfrak{B}[a_{t+1}]$ under the semantic layer L.

Step t is finished.

Based on the process described, construct a computable isomorphism $\mu:\mathcal{B}\to\mathcal{B}'$ as follows. Given an element $c\in\mathcal{B}$ such that

$$c\subseteq\mathbf{1}_0\cup\mathbf{1}_1\cup\ldots\cup\mathbf{1}_k$$

for some k. Determine $\mu(c)$ by the following rule:

$$\mu(c) = \lambda_0^{-1}(c\cap a_0)\cup\mu_0(c\cap b_0)\cup\ldots\cup\lambda_k^{-1}(c\cap a_k)\cup\mu_k(c\cap b_k).$$

In the other case, when

$$-c\subseteq\mathbf{1}_0\cup\mathbf{1}_1\cup\ldots\cup\mathbf{1}_k$$

for some k, we determine $\mu(c)$ by the following rule:

$$\mu(c) = -\big[\lambda_0^{-1}(-c\cap a_0)\cup\mu_0(-c\cap b_0)\cup\ldots\cup\lambda_k^{-1}(-c\cap a_k)\cup\mu_k(-c\cap b_k)\big].$$

The relation [9, (2.2)] characterizes both a particular ultrafilter $\hat{\mathfrak{F}}$ in the direct product $\bigotimes_{i\geqslant 0}\mathcal{B}_i$ and a particular ultrafilter $\hat{\mathfrak{F}}'$ in the direct product $\bigotimes_{i\geqslant -1}\mathcal{B}'_i$. From this, it is simple to prove that so defined mapping μ is indeed a computable isomorphism from $\mathcal{B}$ to $\mathcal{B}'$ and μ preserves the set of model-theoretic properties assigned to the ultrafilters in the summands; moreover, μ maps the ultrafilter $\hat{\hat{\mathfrak{F}}}$ of the first direct product in the ultrafilter $\hat{\hat{\mathfrak{F}}}'$ of the second direct product, and thereby

$$\mu : \mathfrak{B} \to \mathfrak{B}'$$

is a computable isomorphism of the semantic types under the semantic layer L, since the properties of the same theory P are assigned to both $\hat{\hat{\mathfrak{F}}}$ and $\hat{\hat{\mathfrak{F}}}'$.

This ends the proof of Lemma 3.1.

PROOF to Lemma 3.2. For the sake of unification, the semantic type $\mathfrak{B}^*$ is denoted by $\mathfrak{B}'_{-1}$. So, we have two sequences of $\mathcal{E}$-types

$$\mathfrak{B}_i = (\mathcal{B}_i, \nu_i, \xi_i),\ i \in \mathbb{N}, \quad \mathfrak{B}'_i = (\mathcal{B}'_i, \nu'_i, \xi'_i),\ i \in \mathbb{N} \cup \{-1\},$$

satisfying the conditions of Lemma 3.2. Our goal is to construct a computable isomorphism between the following semantic types under M:

$$\mathfrak{B} = \bigotimes\nolimits^{[P]}_{i \geqslant 0} (\mathcal{B}_i, \nu_i, \xi_i) \ \ \textit{and} \ \ \mathfrak{B}' = \bigotimes\nolimits^{[P]}_{i \geqslant -1} (\mathcal{B}'_i, \nu'_i, \xi'_i). \tag{3.4}$$

For this purpose, a step-by-step procedure similar to that given in Lemma 3.1 is performed that determines a sequence of isomorphisms λ_i and μ_i for all $i \in \mathbb{N}$. Scheme in Fig. 3.1 represents main details of the construction. The construction is effective. As a result, a total isomorphism μ between the types (3.4) is obtained.

Thereby, both Lemma 3.1 and Lemma 3.2 are proved. □

Lemma 3.3. *Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a sequence of semantic $\mathcal{F}$-types which is $\mathcal{F}$-rich under a semantic layer $L \subseteq ACL$. There is a general computable function $d(x)$ satisfying*

$$0 = d(0) < d(1) < d(2) < \ldots < d(k) < \ldots$$

such that a new sequence of types $\mathfrak{B}'_t$, $t \in \mathbb{N}$, obtained by a finite gluing operation of adjacent members from the source sequence as follows

$$\mathfrak{B}'_t = \bigotimes\nolimits_{i=d(t)}^{d(t+1)-1} \mathfrak{B}_i, \quad t \in \mathbb{N}, \tag{3.5}$$

consists of $\mathcal{F}$-types and satisfies the following properties:

(a) $\mathfrak{B}'_t$, $t \in \mathbb{N}$, *is a uniformly $\mathcal{F}$-universal sequence under the semantic layer L*,

(b) $\bigotimes^{[P]}_{i \in \mathbb{N}} \mathfrak{B}_i \equiv_L \bigotimes^{[P]}_{t \in \mathbb{N}} \mathfrak{B}'_t$, *for any complete theory P*.

Similar statement for computably axiomatizable types.

Lemma 3.4. *Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a sequence of semantic $\mathcal{E}$-types which is $\mathcal{E}$-rich under a semantic layer $M \subseteq ASL$. There is a general computable function $d(x)$ satisfying*

$$0 = d(0) < d(1) < d(2) < \ldots < d(k) < \ldots$$

such that a new sequence of types $\mathfrak{B}'_t$, $t \in \mathbb{N}$, obtained by a finite gluing operation of adjacent members in the source sequence as follows

$$\mathfrak{B}'_t = \bigotimes\nolimits_{i=d(t)}^{d(t+1)-1} \mathfrak{B}_i, \quad t \in \mathbb{N}, \tag{3.6}$$

consists of $\mathcal{E}$-types and satisfies the following properties:

(a) $\mathfrak{B}'_t$, $t \in \mathbb{N}$, *is a uniformly $\mathcal{E}$-universal sequence under the semantic layer M*,

(b) $\bigotimes^{[P]}_{i\in\mathbb{N}}\mathfrak{B}_i \equiv_M \bigotimes^{[P]}_{t\in\mathbb{N}}\mathfrak{B}'_t$ *for any complete theory* P.

PROOF to Lemma 3.3. Let n_0 be a fixed index of the semantic $\mathcal{F}$-type $\text{₤}(GRE)$ under the semantic layer L. With the help of the function $f(n,x)$ in (1.3) define a function $d(x)$ by the following computable scheme:

$$\begin{cases} d(0)=0, \\ d(t+1)=f(n_0,d(t))+1, \end{cases}$$

and then apply it to obtain a new sequence (3.5). The isomorphism presented in Part (b) is obvious. By construction, the last summand in each finite product (3.5) has a segment $\mathfrak{B}_{d(t+1)-1}[a_t]$, $t\in\mathbb{N}$, whose semantic type is equivalent to $\text{₤}(GRE)$ under the semantic layer L. Thus, any $\mathcal{F}$-type is realized in the restriction $\mathfrak{B}'_t[a_t]$, $t\in\mathbb{N}$, by the RedLev procedure defined in [9, Lem. 1.3]; moreover, the sequence of elements $a_t\in|\mathfrak{B}'_t|$, $t\in\mathbb{N}$, is constructed effectively. This shows that the new obtained sequence of types $\mathfrak{B}'_t$, $t\in\mathbb{N}$, is indeed uniformly $\mathcal{F}$-universal under the semantic layer L.

PROOF to Lemma 3.4. Let n_0 be a fixed index of the semantic type $\mathcal{E}^u_{c.a.} = \text{₤}(T^u_{c.a.})$, cf. [9, (4.1)], under the semantic layer M. With the help of the function $f(n,x)$ in (1.4) define a function $d(x)$ by the following computable scheme:

$$\begin{cases} d(0)=0, \\ d(t+1)=f(n_0,d(t))+1, \end{cases}$$

and then apply it to obtain a new sequence (3.6). The isomorphism presented in Part (b) is obvious. By construction, the last summand in each finite product (3.6) has a segment $\mathfrak{B}_{d(t+1)-1}[a_t]$, $t\in\mathbb{N}$, whose semantic type is equivalent to $\text{₤}(T^\omega_{c.a.})$ under the semantic layer M. Thus, any $\mathcal{E}$-type is realized in the restrictions $\mathfrak{B}'_t[a_t]$, $t\in\mathbb{N}$, according to Lemma 3.2 (b); moreover, the sequence of elements $a_t\in|\mathfrak{B}'_t|$, $t\in\mathbb{N}$, is constructed effectively. This shows that the new obtained sequence of types $\mathfrak{B}'_t$, $t\in\mathbb{N}$, is indeed uniformly $\mathcal{E}$-universal under the semantic layer M.

Both Lemma 3.3 and Lemma 3.4 are proved. □

Theorem 3.5. [1-st Isomorphism Criterion for $\mathcal{F}$-types] *Let* $L\subseteq ACL$ *be a semantic layer,* P *be a complete theory, and the following conditions hold:*

(a) $\mathfrak{B}_i$, $i\in\mathbb{N}$, *is a computable* $\mathcal{F}$*-rich sequence of* $\mathcal{F}$*-types under* L,

(b) $\mathfrak{B}'_j$, $j\in\mathbb{N}$, *is a computable* $\mathcal{F}$*-rich sequence of* $\mathcal{F}$*-types under* L.

Then, $\bigotimes^{[P]}_{i\in\mathbb{N}}\mathfrak{B}_i \equiv_L \bigotimes^{[P]}_{j\in\mathbb{N}}\mathfrak{B}'_j$.

Theorem 3.6. [1-st Isomorphism Criterion for $\mathcal{E}$-types] *Let* $M\subseteq ASL$ *be a semantic layer,* P *be a complete theory, and the following conditions hold:*

(a) $\mathfrak{B}_i$, $i\in\mathbb{N}$, *is a computable* $\mathcal{E}$*-rich sequence of* $\mathcal{E}$*-types under* M,

(b) $\mathfrak{B}'_j$, $j\in\mathbb{N}$, *is a computable* $\mathcal{E}$*-rich sequence of* $\mathcal{E}$*-types under* M.

Then, $\bigotimes^{[P]}_{i\in\mathbb{N}}\mathfrak{B}_i \equiv_M \bigotimes^{[P]}_{j\in\mathbb{N}}\mathfrak{B}'_j$.

PROOF to Theorem 3.5. By Lemma 3.3, one can reorganize the two pointed out $\mathcal{F}$-rich sequences of semantic types in new sequences of types that are uniformly $\mathcal{F}$-universal under the semantic layer L. After that, the required isomorphism is immediately provided by Lemma 3.1. □.

PROOF to Theorem 3.6. By Lemma 3.4, one can reorganize the two pointed out $\mathcal{E}$-rich sequences of semantic types in new sequences of types that are uniformly

$\mathcal{E}$-universal under the semantic layer M. After that, the required isomorphism is immediately provided by Lemma 3.2. $\square$.

One more case when an isomorphism is available.

Theorem 3.7. [2-nd Isomorphism Criterion for $\mathcal{F}$-types] *Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a $\mathcal{F}$-rich computable sequence of $\mathcal{F}$-types under a semantic layer $L \subseteq ACL$, P be a complete theory, and $\mathfrak{B}^*$ be any $\mathcal{F}$-type. Then,*

$$\mathfrak{B}^* \otimes \bigotimes_{i \in \mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_L \bigotimes_{i \in \mathbb{N}}^{[P]} \mathfrak{B}_i. \tag{3.7}$$

The same statement for $\mathcal{E}$-types:

Theorem 3.8. [2-nd Isomorphism Criterion for $\mathcal{E}$-types] *Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a $\mathcal{E}$-rich computable sequence of $\mathcal{E}$-types under a semantic layer $M \subseteq ASL$, P be a complete theory, and $\mathfrak{B}^*$ be any $\mathcal{E}$-type. Then,*

$$\mathfrak{B}^* \otimes \bigotimes_{i \in \mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_M \bigotimes_{i \in \mathbb{N}}^{[P]} \mathfrak{B}_i. \tag{3.8}$$

Proof to Theorem 3.7. Define a new sequence of $\mathcal{F}$-types by the following rule:

$$\mathfrak{B}'_0 = \mathfrak{B}^*,$$
$$\mathfrak{B}'_n = \mathfrak{B}_{n-1}, \textit{ for } n > 0.$$

Obviously, $\mathfrak{B}'_i$, $i \in \mathbb{N}$, is a computable $\mathcal{F}$-rich under M sequence of $\mathcal{F}$-types. Moreover, the left-hand side expression in (3.7) is isomorphic to $\bigotimes_{i \in \mathbb{N}}^{[P]} \mathfrak{B}'_i$ by [10, Lem. 9.11(d)]. Then, it remains to apply Theorem 3.5 to obtain the required isomorphism. $\square$

Proof to Theorem 3.8. We define a new sequence of $\mathcal{E}$-types by the following rule:

$$\mathfrak{B}'_0 = \mathfrak{B}^*,$$
$$\mathfrak{B}'_n = \mathfrak{B}_{n-1}, \textit{ for } n > 0.$$

Obviously, $\mathfrak{B}'_i$, $i \in \mathbb{N}$, is a computable $\mathcal{E}$-rich under the layer L sequence of $\mathcal{E}$-types. Moreover, the left-hand side expression in (3.8) is isomorphic to $\bigotimes_{i \in \mathbb{N}}^{[P]} \mathfrak{B}'_i$ by [10, Lem. 9.11(d)]. Then, it remains to apply Theorem 3.6 to obtain the required isomorphism. $\square$

Exercise 3.9. Establish the following statements:

(a) there is a computable $\mathcal{E}$-rich under ASL sequence of $\mathcal{E}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$, such that $\text{\textit{Ɇ}}(T_{c.a.}^{[EQC]}) \equiv_{ASL} \bigotimes_{i \in \mathbb{N}}^{[EQC^*]} \mathfrak{B}_i$;

(b) $Ɇ(T_{c.a.}^{[EQC]}) \otimes \mathfrak{B} \equiv_{ASL} Ɇ(T_{c.a.}^{[EQC]})$, *for any $\mathcal{E}$-type $\mathfrak{B}$*;

(c) $Ɇ(T_{c.a.}^{u}) \equiv_{ASL} Ɇ(T_{c.a.}^{[EQC]})$;

(d) $Ɇ(T_{c.a.}^{u}) \otimes \mathfrak{B} \equiv_{ASL} Ɇ(T_{c.a.}^{u})$, *for any $\mathcal{E}$-type $\mathfrak{B}$*;

(e) *there is an $\mathcal{E}$-rich computable sequence of $\mathcal{E}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$, under semantic layer ASL such that $Ɇ(T_{c.a.}^{u}) \equiv_{ASL} \bigotimes_{i \in \mathbb{N}}^{[EQC^*]} \mathfrak{B}_i$.*

Hint. (a) Use [10, (9.7)] together with [10, (9.15)].

(b) Use Part (a) to present $Ɇ(T_{c.a.}^{[EQC]})$ by an $\mathcal{E}$-rich sequence of $\mathcal{E}$-types under ASL and then use Theorem 3.8.

(c) Find a such that $Ɇ(T_{c.a.}^{u})[a] \equiv_{ASL} Ɇ(T_{c.a.}^{[EQC]})$; thus, we obtain $Ɇ(T_{c.a.}^{u}) \equiv_{ASL} Ɇ(T_{c.a.}^{[EQC]}) \otimes \mathcal{B}$ for an $\mathcal{E}$-type $\mathfrak{B}$. After that, use statement in Part (b).

(d) Use Part (b) together with Part (c).

(e) Use Part (a) and Part (c).

4 Semantic type of the graph theory

Now, results of Section 9.2 are applied to obtain some characterization of semantic type of the theory *GRE*. Here, main notations from Lemma [9, Lem. 1.3] are used by default.

Lemma 4.1. *There is an $\mathcal{F}$-rich computable sequence of $\mathcal{F}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$, under the semantic layer ACL such that*

$$₤(\mathit{GRE}) \equiv_{ACL} \bigotimes_{i\in\mathbb{N}}^{[GRE_\omega]} \mathfrak{B}_i.$$

Proof. Consider the following new sequence of sentences of the theory *GRE*:

$$\Theta_0 = \neg\theta_2,$$
$$\Theta_i = (\theta_{i+1} \,\&\, \neg\theta_{i+2}), \quad i \in \mathbb{N} \setminus \{0\}.$$

It is easy to check that

$$\mathit{GRE} \cup \{\neg\Theta_i \mid i \in \mathbb{N}\} \vdash \theta_k, \text{ for all } k \in \mathbb{N}.$$

From this, we have got that the set of formulas $\mathit{GRE} \cup \{\neg\Theta_i \mid i \in \mathbb{N}\}$ is a system of axioms of the theory GRE_ω.

One can check that the sequence of sentences Θ_i, $i \in \mathbb{N}$, of the theory *GRE* together with the complete extension GRE_ω of *GRE* exactly meets all the requirements of [10, Lem. 9.1]. Thus we obtain the following decomposition

$$₤(\mathit{GRE}) \equiv_{ACL} \bigotimes_{i\in\mathbb{N}}^{[GRE_\omega]} \mathfrak{B}_i$$

where

$$\mathfrak{B}_0 = ₤(\mathit{GRE} + \Theta_0) = ₤(\mathit{GRE} \cup \{\neg\theta_2\}),$$
$$\mathfrak{B}_k = ₤(\mathit{GRE} + \Theta_k) = ₤(\mathit{GRE} \cup \{\theta_{k+1} \,\&\, \neg\theta_{k+1}\}), \ k \in \mathbb{N} \setminus \{0\}.$$

The fact that this sequence of semantic types is $\mathcal{F}$-rich under the semantic layer *ACL* is provided by [9, Lem. 1.3]. Indeed, let T be a finitely axiomatizable theory and $k \in \mathbb{N}$. Applying the RedLev procedure with a level parameter $e > k+1$, we obtain a pair of objects

$$(F, I) = \mathrm{RedLev}(T, e),$$

where F is a finitely axiomatizable extension of the theory

$$\mathit{GRE} \cup \{\theta_e, \neg\theta_{e+1}\} = \mathit{GRE} \cup \{\Theta_{e-1}\}, \quad e-1 > k.$$

Therefore, finitely axiomatizable semantic type $₤(T)$ under the semantic layer *ACL* is equivalent to a segment $\mathfrak{B}_{e-1}[F]$ of the type $\mathfrak{B}_{e-1}$; moreover, corresponding computable isomorphism is found effectively.

This gives the required statement. □

Lemma 4.2. *We have $₤(\mathit{GRE}) \equiv_{ACL} \bigotimes_{i\in\mathbb{N}}^{[GRE_\omega]} \mathfrak{B}_i$, for any computable rich under ACL sequence of $\mathcal{F}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$.*

Proof. It is a direct consequence of Theorem 3.5 and Lemma 4.1. □

Lemma 4.3. $₤(\mathit{GRE}) \otimes \mathfrak{B} \equiv_{ACL} ₤(\mathit{GRE})$ *for any $\mathcal{F}$-type $\mathfrak{B}$.*

Proof. By virtue of Lemma 4.2, $\text{₤}(GRE)$ is equivalent to the direct product of a computable rich under ACL sequence of $\mathcal{F}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$. By Theorem 3.7, we have

$$\text{₤}(GRE) \otimes \mathfrak{B} \equiv_{ACL} \mathfrak{B} \otimes \text{₤}(GRE) \equiv_{ACL} \text{₤}(GRE),$$

that gives the required statement. □

As an important consequence, we obtain:

Lemma 4.4. $\text{₤}(GR) \equiv_{ACL} \text{₤}(GRE)$.

Proof. Let Θ be a sentence of the signature $\sigma = \sigma_{GR} = \{\Gamma^2\}$ which is an axiom of GRE. Then, $GRE = [GR \cup \{\Theta\}]^\sigma$ holds. Consider finitely axiomatizable theory $T = [GR \cup \{\neg\Theta\}]^\sigma$. By [10, Lem. 8.1], we have the following algebraic isomorphism

$$GR\langle c\rangle \approx_a \left[\, GR \cup \{\Theta\}\right]^\sigma \otimes \left[\, GR \cup \{\neg\Theta\}\right]^\sigma \approx_a GRE \otimes T.$$

Therefore, we have

$$\text{₤}(GR) \equiv_{ACL} \text{₤}(GRE) \otimes \text{₤}(T).$$

From this, we obtain $\text{₤}(GR) \equiv_{ACL} \text{₤}(GRE)$ by virtue of Lemma 4.3. □

Remark 4.5. The result obtained shows that either of the theories GR or GRE can be considered in the paper instead of the other one. The only particular point exists where the theory GRE must be taken. Namely, an input theory for the universal construction has to be the graph theory GRE, since these requirements are important in the construction. This is an explanation why we use theory GRE instead of a simpler theory GR.

5 Theories suitable for the particular ultrafilter

Introduce a technical definition.

Definition 4.A. Let D be a semantic layer (a default value ACL is supposed for D whenever it is not specified explicitly). A theory P of a finite signature σ is said to be *f-dense* under the semantic layer D if the following properties are satisfied:

(a) theory P is complete and decidable,

(b) for any $\Phi \in SL(\sigma)$ satisfying $P \vdash \Phi$, a sentence $\Psi \in SL(\sigma)$ and a computable isomorphism μ can be found, satisfying the following properties: $P \vdash \neg\Psi$, $\vdash \Psi \to \Phi$, and $\text{₤}([\Psi]^\sigma) \equiv_D \text{₤}(GRE)$ by means of μ; moreover, both a Gödel number of Ψ and an index of the isomorphism μ are found effectively from a Gödel number of the sentence Φ.

The following property takes place:

Lemma 5.1. *Let a theory P be f-dense under a semantic layer $L \subseteq ACL$. Then, P is not finitely axiomatizable.*

Proof. For contrary, suppose that the theory P is finitely axiomatizable, and let Φ be a sentence which is its axiom. The theory P is complete by definition. Then, we have $P \vdash \Phi$, but none sentence Ψ is possible with the properties stated above, because Φ determines a complete theory, while Ψ should determine some of its extensions, which must be a non-complete theory by definition. □

Exercise 5.2. *Let a theory P be f-dense under a semantic layer ACL. Then, for any sentence Φ satisfying $P \vdash \Phi$, Φ has a finite model.*

HINT. Model-theoretic property $\mathfrak{p} =$ "theory has a finite model" belongs to the semantic layer *ACL*.

Compare the concept of *f*-density with that of *inf*-density given in [9, Def. 7.B].

Lemma 5.3 [$\mathbb{U}$]. *Let a theory P be f-dense under a semantic layer L. Then, P is an inf-dense theory under any semantic layer $K \subseteq MQL \cap L$.*

PROOF. This fact is a routine consequence of the two definitions. One can check that the definition of an *f*-dense theory is obtained by way of strengthen of some internal parts in the definition of an *inf*-dense theory. A finite signature is considered instead of an enumerable signature. A finite (or even empty) set of sentences is considered instead of a computable frame Σ. Apart from that, we can apply the universal construction that provides the existence of an equivalence $\text{\textsterling}(T^{u}_{c.a.}) \equiv_{MQL} GRE[\Theta]$ which is a similarity relation under the semantic layer *MQL*, where $GRE[\Theta]$ is an appropriate finitely axiomatizable extension of the graph theory *GRE*. As a result, we obtain that the theory P must be *inf*-dense under any semantic layer $K \subseteq MQL \cap L$. □

Notice a simple fact.

Lemma 5.4 *The following statements hold*:

(a) *Let P be an f-dense theory under a semantic layer L, and $K \subseteq L$. Then, the theory P is f-dense under the semantic layer K.*

(b) *Let R be an inf-dense theory under a semantic layer M, and $N \subseteq M$. Then, the theory R is inf-dense under the semantic layer N.*

PROOF. Immediately, from the definitions. □

Give an example of an *f*-dense theory.

Lemma 5.5. *Theory GRE_ω is f-dense under any semantic layer $L \subseteq ACL$.*

PROOF. Check that the theory GRE_ω is *f*-dense under the semantic layer *ACL*. By Part (d) of [9, Lem. 1.3], the theory GRE_ω is complete and decidable. Now, let Φ be a sentence such that $GRE_\omega \vdash \Phi$. By Compactness Theorem, we obtain $GRE \cup \{\theta_1, \ldots, \theta_k\} \vdash \Phi$ for some finite k. By Part (a) of Lemma [9, Lem. 1.3], we obtain for this k, that $GRE \cup \{\theta_k\} \vdash \Phi$. From this, we have $GRE \cup \{\theta_k, \neg\theta_{k+1}\} \vdash \Phi$; that is, $GRE_k \vdash \Phi$. Take a sentence a such that $\text{\textsterling}(GRE) \equiv_{ACL} \text{\textsterling}(GRE_k)[a]$. We can find such an element a effectively by the RedLev procedure, applying it to the theory *GRE* with a level parameter $e = k$. After that, we put $\Psi = GRE_k \cup \{a\}$. Based on the fact that $GRE_k \vdash \neg\theta_{k+1}$, and therefore, $\Psi \vdash \neg\theta_{k+1}$, we obtain all the properties required in the definition of an *f*-dense theory. It is proved that GRE_ω is an *f*-dense theory under the semantic layer *ACL*; then, it is *f*-dense under any smaller semantic layer $L \subseteq ACL$. □

Consider examples of *f*-dense and *inf*-dense theories.

EXAMPLE 5.6. The theory GRE_ω is both *f*-dense under the semantic layer *ACL* and *inf*-dense under the semantic layer *MQL*. This fact follows from Lemma 5.3 and Lemma 5.5.

EXAMPLE 5.7. Consider theories EQC and EQC^* introduced in the work [10, Sec. 9]. Theory EQC is obviously neither *f*-dense nor *inf*-dense. As for theory EQC^*, it satisfies all *f*-density demands under the semantic layer $ASL \supseteq ACL$ excepting for the requirement of having a finite signature. Thus, EQC^* is *inf*-dense under the semantic layer ASL.

EXERCISE 5.8. Let L be a subset of *ACL*, $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a computable $\mathcal{F}$-rich sequence of $\mathcal{F}$-types under L, and P be an arbitrary complete theory. Suppose that the product of semantic types

$$\mathfrak{B} = \bigoplus_{i<\omega}^{[P]} \mathfrak{B}_i \tag{5.1}$$

is an $\mathcal{F}$-type under L. This means that there is a finitely axiomatizable theory F of a finite signature σ such that $\mathfrak{B} \equiv_L \text{\textsterling}(F)$; i.e., there is a computable isomorphism $\mu : \mathfrak{B} \to \text{\textsterling}(F)$ preserving the layer L. Consider a particular ultrafilter $\hat{\mathfrak{F}}$ in the product (5.1). Let F^* be a complete extension of F that is an μ-image of $\hat{\mathfrak{F}}$. Prove that the following conditions are satisfied:

(a) F^* is a complete and decidable theory,

(b) F^* is an *f*-dense theory,

(c) $P \equiv_L F^*$.

Moreover, a characteristic index of F^* is found effectively in a c.e. index of the sequence $\mathfrak{B}_i$, $i \in \mathbb{N}$, Gödel number of F, and c.e. index of the isomorphism μ.

Hint. Theory F^* is complete and decidable because this theory is an μ-image of a computable filter $\hat{\mathfrak{F}}$ in semantic type (5.1) by virtue of [10, Lem. 2.2]. Based on the isomorphism μ, we obtain a powerful tool to study properties of theory F^* by way of passing to μ-preimages of sentences and manipulating with corresponding elements in the direct product (5.1). moreover, we have that for each sentence Φ of signature σ, if $F^* \vdash \Phi$ then $\mu^{-1}(\Phi) \in \hat{\mathfrak{F}}$, i.e., this element has the form [10, (2.3)(a)], while in the other case $F^* \nvdash \Phi$, $\mu^{-1}(\Phi) \notin \hat{\mathfrak{F}}$, i.e., this element has the form [10, (2.3)(b)]. By applying this method, we can easily verify that all requirements of the definition of *f*-density for the theory F^* are satisfied. Indeed, let Φ be a sentence of signature σ such that $F^* \vdash \Phi$. Then, $e = \mu^{-1}(\Phi)$ satisfies [10, (2.3)(a)]; i.e., this element covers all members $\mathfrak{B}_i$, $i \geqslant k_0$, for some $k_0 > 0$. Applying the fact that sequence $\mathcal{B}_i$, $i \in \mathbb{N}$, is $\mathcal{F}$-rich under L, we can effectively find $i > k_0$ and an element a in $\mathfrak{B}_i$ such that $c = \mathfrak{B}[a] \equiv_L \text{\textsterling}(\mathit{GRE})$. Thereby, we found an element $c \subseteq e$ such that $c \notin \hat{\mathfrak{F}}$ and $\mathfrak{B}[c] \equiv_L \text{\textsterling}(\mathit{GRE})$. Let $\Psi = \mu(c)$. Since μ is an isomorphism of semantic types, we obtain finally $F^* \nvdash \Psi$, $\Psi \vdash \Phi$, and $\text{\textsterling}(\Psi) \equiv_L \text{\textsterling}(\mathit{GRE})$ thus confirming that theory F^* is indeed an *f*-dense theory. Part (c) is ensured by the fact that both theories P and F^* are complete and decidable and by construction have identical model-theoretic properties under layer L. Part (c) is ensured by definitions. Effectiveness is an immediate consequence of the construction. □

The case of $\mathcal{E}$-types leads to an analogous situation.

Exercise 5.9. Let M be a subset of *MQL*, $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a computable $\mathcal{E}$-rich sequence of $\mathcal{E}$-types under M, and R be an arbitrary complete theory. Suppose that the product of semantic types

$$\mathfrak{B} = \bigoplus_{i<\omega}^{[R]} \mathfrak{B}_i \tag{5.2}$$

is an $\mathcal{E}$-type under M. This means that there is a computably axiomatizable theory T of an enumerable signature σ such that $\mathfrak{B} \equiv_M \text{\textsterling}(F)$; i.e., there is a computable isomorphism $\mu : \mathfrak{B} \to \text{\textsterling}(F)$ preserving the layer M. Consider a particular ultrafilter $\hat{\mathfrak{F}}$ in the product (5.2). Let T^* be a complete extension of T that is an μ-image of $\hat{\mathfrak{F}}$. Prove that the following conditions are satisfied:

(a) T^* is a complete and decidable theory,

(b) T^* is an *inf*-dense theory,

(c) $R \equiv_M T^*$.

Moreover, a characteristic index of T^* is found effectively in c.e. indices of the sequence $\mathfrak{B}_i$, $i \in \mathbb{N}$, theory T, and the isomorphism μ.

Hint. By the same scheme as in Exercise 5.8 by replacing the concept of $\mathcal{F}$-type by that of $\mathcal{E}$-type and elements of the definition of an *f*-dense theory by corresponding elements of the definition of an *inf*-dense theory. □

Remark 5.10. Two given above exercises demonstrate essence of the idea of concepts of an *f*-dense and an *inf*-dense theory.

In this section, we apply methods of previous sections to assemble universal semantic types for both finitely axiomatizable and computably axiomatizable cases. This will gives us characterization of the universal semantic types in both cases.

6 Universal finitely axiomatizable semantic types

In this section, we characterize finitely axiomatizable semantic types that are universal under the class of all such objects. Here, only $\mathcal{F}$-types are considered.

We start from a thin property of an *f*-dense theory:

Lemma 6.1. *Given a semantic layer $D \subseteq ACL$. Let P be a complete theory of a finite signature σ that is f-dense under D. There is a computable $\mathcal{F}$-rich under D sequence of $\mathcal{F}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}$, such that the following semantic type*

$$\mathfrak{B} = \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i$$

is an $\mathcal{F}$-type under the semantic layer D.

Proof. The idea of constructing a finitely axiomatizable theory having such a semantic type is based on the method of repeated application of the effectiveness property in the definition of an *f*-dense theory, cf. Definition 4.A.

First of all, fix a sentence Ω of signature σ such that $P \vdash \Omega$.

Let Φ_n, $n \in \mathbb{N}$, be an effective numbering of the set of all sentences Φ in $SL(\sigma)$ such that $P \vdash \neg\Phi$. Such a numbering exists because P is a decidable theory by the definition of an *f*-dense theory. By steps, we construct a new sequence of sentences Ψ_i, $i \in \mathbb{N}$ of signature σ satisfying $P \vdash \neg\Psi_i$ for all i.

Description of the construction:

Step 0. Find a sentence Ψ_0 such that

$$P \vdash \neg\Psi_0, \quad \vdash \Psi_0 \to \Omega, \quad ₤(\Psi_0) \equiv_D ₤(GRE).$$

Since P is supposed to be an *f*-dense theory under D, such a sentence Ψ_0 should exist by definition.

Let $t-1$ steps be already passed, and as a result, a sequence of sentences $\Psi_0, \Psi_1, \dots, \Psi_{t-1}$ is already constructed; moreover, $P \vdash \neg\Psi_i$ for $i = 0, 1, \dots, t-1$.

Step $t > 0$. By definition of an *f*-dense property, we can effectively find a sentence Ψ'_t of signature σ such that

$$\begin{aligned}
&P \vdash \neg\Psi'_t,\\
&\vdash \Psi'_t \to [\Omega \,\&\, \neg(\Psi_0 \vee \Psi_1 \vee \dots \vee \Psi_{t-1})],\\
&₤(\Psi'_t) \equiv_D ₤(GRE).
\end{aligned}$$

After that, we set

$$\Psi_t = \Omega \,\&\, (\Psi_t' \vee \Phi_0 \vee \Phi_1 \vee \ldots \vee \Phi_{t-1}) \,\&\, \neg(\Psi_0 \vee \Psi_1 \vee \ldots \vee \Psi_{t-1}).$$

One can check that the condition $P \vdash \neg\Psi_t$ is satisfied.

Step t is finished.

Description of the construction by steps is complete.

One can check that the following properties of the sequence of sentences Ψ_i, $i \in \mathbb{N}$, constructed for ω steps are satisfied:

Reference_Block (6.1)

(a) $P = [\{\Omega\} \cup \{\neg\Phi_0, \neg\Phi_1, \ldots, \neg\Phi_t, \ldots\}]^\sigma$,

(b) $\vdash \big[\Omega \,\&\, (\Phi_0 \vee \Phi_1 \vee \ldots \vee \Phi_{t-1})\big] \to (\Psi_0 \vee \Psi_1 \vee \ldots \vee \Psi_t)$, *for all* $t \in \mathbb{N}$,

(c) $\vdash (\Psi_t' \to \Psi_t) \,\&\, (\Psi_t \to \Omega)$, *for all* $t \in \mathbb{N}$,

(d) $P \vdash \neg\Psi_t$, *for all* $t \in \mathbb{N}$,

(e) $\vdash \Psi_i \to \neg\Psi_j$, *for all* $i, j \in \mathbb{N}$ *such that* $i \neq j$,

(f) $P = [\{\Omega\} \cup \{\neg\Psi_0, \neg\Psi_1, \ldots, \neg\Psi_t, \ldots\}]^\sigma$,

(g) $(\forall t \in \mathbb{N})(\exists a)\Big(\pounds(GRE) \equiv_D \pounds([\Psi_t]^\sigma)[a]\Big)$.

End_Ref

Now, we turn to the final part of the proof of Lemma 6.1. Denote $\mathfrak{B}_i = \pounds(\Psi_i)$, $i \in \mathbb{N}$. By the properties (6.1)(e,f) the sequence of sentences Ψ_i, $i \in \mathbb{N}$, represents a partition of the unit element, and the set of their negations together with Ω determines the theory P. From this, by [10, Lem. 9.2], we have:

$$\pounds([\Omega]^\sigma) = \bigotimes\nolimits_{i\in\mathbb{N}}^{[P]} \pounds([\Psi_i]^\sigma) = \bigotimes\nolimits_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i.$$

By our construction, $\pounds([\Psi_i]^\sigma)$, $i \in \mathbb{N}$, is a computable sequence of $\mathcal{F}$-types, and it is $\mathcal{F}$-rich under the semantic layer D by virtue of the property (6.1)(g). Moreover, the semantic type $\pounds([\Omega]^\sigma)$ is an $\mathcal{F}$-type since it is defined by a single sentence Ω of a finite signature σ.

Lemma 6.1 is completely proved. □

Lemma 6.2. *Given semantic layers* L, D, *and* N, *such that* $L \subseteq D \cap N$ *and* $D \cup N \subseteq ACL$. *Let* $\mathfrak{B}_i$, $i \in \mathbb{N}$, *be an arbitrary computable* $\mathcal{F}$*-rich sequence of* $\mathcal{F}$*-types under the semantic layer* N, *and* P *be an f-dense theory under* D. *Then, the following semantic type*

$$\mathfrak{B} = \bigotimes\nolimits_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i$$

is an $\mathcal{F}$*-type under the semantic layer* L.

Proof. By Lemma 1.5(a), sequence $\mathfrak{B}_i$, $i \in \mathbb{N}$, is $\mathcal{F}$-rich under L, while by Lemma 5.4(a), theory P is f-dense under L. By applying Lemma 6.1, we find a computable $\mathcal{F}$-rich sequence $\mathfrak{B}_i'$, $i \in \mathbb{N}$, such that $\mathfrak{B}' \equiv_L \bigotimes\nolimits_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i'$ is a finitely axiomatizable type. Finally, by applying Theorem 3.5, we deduce that $\mathfrak{B}' \equiv_L \mathfrak{B}$; thereby, $\mathfrak{B}$ is an $\mathcal{F}$-type under L. □

Lemma 6.3. *Given semantic layers* L, D, *and* N, *such that* $L \subseteq D \cap N$ *and* $D \cup N \subseteq ACL$. *Let* $\mathfrak{B}_i$, $i \in \mathbb{N}$, *be a computable sequence of* $\mathcal{F}$*-types that is* $\mathcal{F}$*-rich under* N. *Then, for any two f-dense theories* P *and* R *under the layer* D, *we have*

$$\bigotimes\nolimits_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_L \bigotimes\nolimits_{j\in\mathbb{N}}^{[R]} \mathfrak{B}_i.$$

Proof. Since $\mathfrak{B}_i$, $i \in \mathbb{N}$, is a computable $\mathcal{F}$-rich under N sequence of $\mathcal{F}$-types, by Lemma 1.5(a), it is $\mathcal{F}$-rich under the layer $L \subseteq N$. By condition, both P and R

are f-dense under D. By Lemma 5.4(a) we obtain that these theories are f-dense under the layer $L \subseteq D$. Consider two following semantic types under the layer L:

$$\mathfrak{B} = \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i, \quad \mathfrak{B}' = \bigotimes_{i\in\mathbb{N}}^{[R]} \mathfrak{B}_i. \tag{6.2}$$

By Lemma 6.2, both $\mathfrak{B}$ and $\mathfrak{B}'$ are $\mathcal{F}$-types under the semantic layer L. By definition of an $\mathcal{F}$-rich under L sequence of $\mathcal{F}$-types, the type $\mathfrak{B}$ presented in (6.2), is equivalent under L to $\mathfrak{B}_k[a]$ for some k and a. Therefore, we have the following chain of similarities under the semantic layer L:

$$\begin{aligned}
&\mathfrak{B}' \equiv_L \bigotimes_{i\in\mathbb{N}}^{[R]} \mathfrak{B}_i \equiv_L \mathfrak{B}_k \otimes \bigotimes_{i\in\mathbb{N}\smallsetminus\{k\}}^{[R]} \mathfrak{B}_i \equiv_L \\
&\Big(\mathfrak{B}_k[a] \otimes \mathfrak{B}_k[-a]\Big) \otimes \bigotimes_{i\in\mathbb{N}\smallsetminus\{k\}}^{[R]} \mathfrak{B}_i \equiv_L \\
&\mathfrak{B}_k[a] \otimes \Big(\mathfrak{B}_k[-a] \otimes \bigotimes_{i\in\mathbb{N}\smallsetminus\{k\}}^{[R]} \mathfrak{B}_i\Big) \equiv_L \\
&\mathfrak{B} \otimes \mathfrak{B}'' \equiv_L \mathfrak{B}'' \otimes \mathfrak{B} \equiv_L \\
&\mathfrak{B}'' \otimes \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \overset{\text{(by Theorem 3.7)}}{\equiv_L} \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_L \mathfrak{B},
\end{aligned}$$

where $\mathfrak{B}''$ is a notation to the type in brackets from the preceding line. Notice that, the sequence of $\mathcal{F}$-types $\mathfrak{B}_i$, $i \in \mathbb{N}\smallsetminus\{k\}$, is obviously computable and $\mathcal{F}$-rich under L. Therefore, by Lemma 6.2, the type $\bigotimes_{i\in\mathbb{N}\smallsetminus\{k\}}^{[R]} \mathfrak{B}_i$ is an $\mathcal{F}$-type, and by [10, Lem. 7.1], $\mathfrak{B}[-a]$ is also an $\mathcal{F}$-type. Then, by Lemma 6.2 together with [10, Lem. 10.1], $\mathfrak{B}''$ is an $\mathcal{F}$-type under the semantic layer L, and this ensures that Theorem 3.7 is indeed applicable to this situation. □

Now, we characterize semantic type of the theory *GRE*.

Lemma 6.4. *Given semantic layers L, D, and N, such that $L \subseteq D \cap N$ and $D \cup N \subseteq ACL$. Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be an arbitrary computable $\mathcal{F}$-rich sequence of $\mathcal{F}$-types under the semantic layer N, and P be an f-dense theory under D. Then, the following isomorphism takes place*:

$$₤(GRE) \equiv_L \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i.$$

Proof. By Lemma 4.1, we have $₤(GRE) \equiv_{ACL} \bigotimes_{i\in\mathbb{N}}^{[GRE_\omega]} \mathfrak{B}'_i$ for an $\mathcal{F}$-rich under *ACL* computable sequence of $\mathcal{F}$-types with an f-dense under *ACL* theory GRE_ω. By Lemma 1.5(a), from $L \subseteq N$, we obtain that the sequence $\mathfrak{B}'_i$, $i \in \mathbb{N}$, is also $\mathcal{F}$-rich under the semantic layer L. By Lemma 5.4(a), GRE_ω is an f-dense theory under the layer $L \subseteq D$. By applying Lemma 3.5 and then Lemma 6.3, we obtain

$$\bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_L \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}'_i \equiv_L \bigotimes_{i\in\mathbb{N}}^{[GRE_\omega]} \mathfrak{B}'_i \equiv_{ACL} ₤(GRE),$$

that is exactly what is required. □

Now, we formulate a general statement characterizing universal finitely axiomatizable types under the algebraic Cartesian semantic layer *ACL* as well as under any smaller semantic layer.

Theorem 6.5. *Given semantic layers L, D, and N, such that $L \subseteq D \subseteq ACL$ and $L \subseteq N \subseteq ACL$. For any abstract semantic type $\mathfrak{B}$ the following assertions are equivalent with each other*:

(a) *$\mathfrak{B}$ is an $\mathcal{F}$-type and is $\mathcal{F}$-universal under the semantic layer L,*

(b) *$\mathfrak{B}$ is an $\mathcal{F}$-type and is weak $\mathcal{F}$-universal under the semantic layer L,*

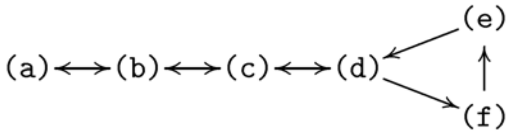
(c) $\mathfrak{B}$ *is an* $\mathcal{F}$*-type and* $(\exists a)\Big(\mathfrak{B}[a] \equiv_L ₤(\mathit{GRE})\Big)$,

(d) $\mathfrak{B} \equiv_L ₤(\mathit{GRE})$,

(e) $\mathfrak{B} \equiv_L \bigotimes_{k\in\mathbb{N}}^{[P]} \mathfrak{B}_k$, *for an f-dense under D theory P and a computable* $\mathcal{F}$*-rich under N sequence of* $\mathcal{F}$*-types* $\mathfrak{B}_k$, $k \in \mathbb{N}$.

(f) $\mathfrak{B} \equiv_L \bigotimes_{k\in\mathbb{N}}^{[P]} \mathfrak{B}_k$, *for any f-dense under D theory P and any computable* $\mathcal{F}$*-rich under N sequence of* $\mathcal{F}$*-types* $\mathfrak{B}_k$, $k \in \mathbb{N}$.

Proof. We prove equivalences among the parts by the following scheme:



(a)⇔(b) Provided by Lemma 1.3(a).

(b)⇔(c) Implication (b)⇒(c) follows from the definition of a weak universal semantic type, cf. Section 1. The back implication (c)⇒(b) is a consequence of the universality property stated in Lemma 1.6 for the semantic type $₤(\mathit{GRE})$ together with transitivity of the relation of being a segment of an algebra.

(c)⇔(d) Part ⇒ is followed from Lemma 4.3 and [10, Lem. 10.1(e)], while the back implication is obvious.

(d)⇒(f) Given a semantic type $\mathfrak{B} = \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i$ satisfying properties pointed out in (f). By Lemma 4.1, we have $₤(\mathit{GRE}) \equiv_{ACL} \bigotimes_{i\in\mathbb{N}}^{[GRE_\omega]} \mathfrak{B}'_i$ for an $\mathcal{F}$-rich under ACL computable sequence of $\mathcal{F}$-types. By Lemma 1.5(a), the sequence $\mathfrak{B}'_i$, $i \in \mathbb{N}$, is an $\mathcal{F}$-rich under L sequence of $\mathcal{F}$-types. By Lemma 5.4(a), GRE_ω is an f-dense theory under the layer $L \subseteq ACL$. By applying Lemma 3.5 and then Lemma 6.3, we obtain

$$\mathfrak{B} = \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_L \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}'_i \equiv_L \bigotimes_{i\in\mathbb{N}}^{[GRE_\omega]} \mathfrak{B}'_i \equiv_{ACL} ₤(\mathit{GRE}),$$

that is exactly what is required.

(f)⇒(e) The implication is obvious whenever we establish that there is at least one $\mathcal{F}$-rich under N sequence of $\mathcal{F}$-types and an f-dense under D theory. By Lemma 5.5, theory GRE_ω is f-dense under ACL. By applying Lemma 4.1 together with Lemma 1.5(a) and Lemma 5.4(a), we obtain that the pointed out sequence and an f-dense theory indeed exist.

(e)⇒(d) It is provided by Lemma 6.4.

Theorem 6.5 is proved. □

Theorem 6.5 represents an advanced and extended version of the main statement of the paper [2, Th. 7.1].

7 Universal computably axiomatizable semantic types

In this section, we characterize computably axiomatizable semantic types that are universal under all such objects. Here, only $\mathcal{E}$-types are considered.

We start from a principal property of an *inf*-dense theory:

Lemma 7.1. *Given a semantic layer* $E \subseteq ASL$. *Let R be a complete theory of an enumerable signature* σ *that is inf-dense under E. There is a computable* $\mathcal{E}$*-rich under E sequence of* $\mathcal{E}$*-types* $\mathfrak{B}_i$, $i \in \mathbb{N}$, *such that, the following semantic type*

$$\mathfrak{B} = \bigotimes_{i\in\mathbb{N}}^{[R]} \mathfrak{B}_i$$

is an $\mathcal{E}$-type under the semantic layer E.

PROOF. By [10, Th. 6.2(c)], the sequence of semantic $\mathcal{E}$-types [10, (6.2)] is computable and $\mathcal{E}$-representative. On the other hand, any $\mathcal{E}$-representative under K sequence of $\mathcal{E}$-types is $\mathcal{E}$-rich under the layer K. By applying [10, Lem. 9.11], we obtain exactly what is required.

Lemma 7.1 is proved. □

Lemma 7.2. *Given semantic layers K, E, and M, such that $K \subseteq E \cap M$ and $E \cup M \subseteq ASL$. Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be an arbitrary computable $\mathcal{E}$-rich sequence of $\mathcal{E}$-types under the semantic layer M, and R be an inf-dense theory under E. Then, the following semantic type*

$$\mathfrak{B} = \bigotimes_{i\in\mathbb{N}}^{[R]} \mathfrak{B}_i$$

is an $\mathcal{E}$-type under the semantic layer K.

PROOF. We obtain this statement by [10, Lem. 9.11], also applying statements of Lemma 1.5(b) and Lemma 5.4(b). □

Lemma 7.3. *Given semantic layers K, E, and M, such that $K \subseteq E \cap M$ and $E \cup M \subseteq ASL$. Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be a computable sequence of $\mathcal{E}$-types that is $\mathcal{E}$-rich under M. Then, for any two inf-dense theories P and R under the layer E, we have*

$$\bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_K \bigotimes_{j\in\mathbb{N}}^{[R]} \mathfrak{B}_i.$$

PROOF. Consider two following semantic types

$$\mathfrak{B} = (\mathcal{B}, \nu, p) = \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i, \quad \mathfrak{B}' = (\mathcal{B}', \nu', p') = \bigotimes_{i\in\mathbb{N}}^{[R]} \mathfrak{B}_i. \tag{7.1}$$

By Lemma 7.2, both $\mathfrak{B}$ and $\mathfrak{B}'$ are $\mathcal{E}$-types under the semantic layer K. By definition of a $\mathcal{E}$-rich under a semantic layer sequence of $\mathcal{E}$-types, the type $\mathfrak{B}$ itself pointed out in (7.1) is equivalent to $\mathfrak{B}_k[a]$ for some k and a. Therefore, we have the following chain of isomorphisms under the semantic layer $K \subseteq ASL$:

$$\begin{aligned}
&\mathfrak{B}' \equiv_K \bigotimes_{i\in\mathbb{N}}^{[R]} \mathfrak{B}_i \equiv_K \mathfrak{B}_k \otimes \bigotimes_{i\in\mathbb{N}\smallsetminus\{k\}}^{[R]} \mathfrak{B}_i \equiv_K \\
&\Big(\mathfrak{B}_k[a] \otimes \mathfrak{B}_k[-a]\Big) \otimes \bigotimes_{i\in\mathbb{N}\smallsetminus\{k\}}^{[R]} \mathfrak{B}_i \equiv_K \\
&\mathfrak{B}_k[a] \otimes \Big(\mathfrak{B}_k[-a] \otimes \bigotimes_{i\in\mathbb{N}\smallsetminus\{k\}}^{[R]} \mathfrak{B}_i\Big) \equiv_K \\
&\mathfrak{B} \otimes \mathfrak{B}'' \equiv_K \mathfrak{B}'' \otimes \mathfrak{B} \equiv_K \\
&\mathfrak{B}'' \otimes \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \overset{\text{(by Theorem 3.8)}}{\equiv_K} \bigotimes_{i\in\mathbb{N}}^{[P]} \mathfrak{B}_i \equiv_K \mathfrak{B},
\end{aligned}$$

where $\mathfrak{B}''$ denotes a type in the brackets at a preceding line. It is possible to check that $\mathfrak{B}''$ is an $\mathcal{E}$-type under the semantic layer K, ensuring that Theorem 3.8 is indeed applicable to this situation. □

Now, we characterize semantic type of the theory $T^u_{c.a.}$.

Lemma 7.4. *Given semantic layers K, E, and M, such that $K \subseteq E \cap M$ and $E \cup M \subseteq ASL$. Let $\mathfrak{B}_i$, $i \in \mathbb{N}$, be an arbitrary computable $\mathcal{E}$-rich sequence of $\mathcal{E}$-types under the semantic layer M, and R be an inf-dense theory under E. Then, the following isomorphism takes place*:

$$\text{₤}(T^u_{c.a.}) \equiv_K \bigotimes_{i\in\mathbb{N}}^{[R]} \mathfrak{B}_i.$$

Proof. By Exercise 3.9(e), we have $\pounds(T^u_{c.a.}) \equiv_{ASL} \bigotimes^{[EQC^*]}_{i\in\mathbb{N}} \mathfrak{B}'_i$ for an $\mathcal{E}$-rich under ASL computable sequence of $\mathcal{E}$-types with an *inf*-dense under ASL theory EQC^*. By Lemma 1.5(b), from $K \subseteq ASL$, we obtain that the sequence $\mathfrak{B}'_i$, $i\in\mathbb{N}$, is also $\mathcal{E}$-rich under the semantic layer K. By [10, Lem. 9.14], EQC^* is an *inf*-dense theory under the layer $E \subseteq ASL$. By applying Lemma 3.6 and then Lemma 7.3, we obtain

$$\bigotimes^{[R]}_{i\in\mathbb{N}} \mathfrak{B}_i \equiv_K \bigotimes^{[R]}_{i\in\mathbb{N}} \mathfrak{B}'_i \equiv_K \bigotimes^{[EQC^*]}_{i\in\mathbb{N}} \mathfrak{B}'_i \equiv_{ASL} \pounds(T^u_{c.a.}),$$

that is exactly what is required. □

A comprehensive characterization of universal $\mathcal{E}$-types is also possible, similar to that given earlier for $\mathcal{E}$-types:

Theorem 7.5. *Given semantic layers K, E, and M, such that $K \subseteq E \subseteq MQL$ and $K \subseteq M \subseteq MQL$. For any abstract semantic type $\mathfrak{B}$ the following assertions are equivalent with each other*:

(a) $\mathfrak{B}$ *is an $\mathcal{E}$-type and is $\mathcal{E}$-universal under the semantic layer K,*

(b) $\mathfrak{B}$ *is an $\mathcal{E}$-type and is weak $\mathcal{E}$-universal under the semantic layer K,*

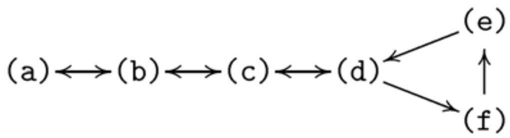
(c) $\mathfrak{B}$ *is an $\mathcal{E}$-type and* $(\exists a)\Big(\mathfrak{B}[a] \equiv_K \pounds(T^u_{c.e.})\Big)$,

(d) $\mathfrak{B} \equiv_K \pounds(T^u_{c.e.})$,

(e) $\mathfrak{B} \equiv_K \bigotimes^{[R]}_{k\in\mathbb{N}} \mathfrak{B}_k$, *for an inf-dense under E theory R and a computable $\mathcal{E}$-rich under M sequence of $\mathcal{E}$-types $\mathfrak{B}_k$, $k\in\mathbb{N}$.*

(f) $\mathfrak{B} \equiv_K \bigotimes^{[R]}_{k\in\mathbb{N}} \mathfrak{B}_k$, *for any inf-dense under E theory R and any computable $\mathcal{E}$-rich under M sequence of $\mathcal{E}$-types $\mathfrak{B}_k$, $k\in\mathbb{N}$.*

Proof. We prove equivalences among the parts by the following scheme:



(a)⇔(b) Provided by Lemma 1.3(b).

(b)⇔(c) Implication (b)⇒(c) follows from the definition of weak universality for semantic $\mathcal{E}$-types, cf. Section 1. The back implication (c)⇒(b) is a consequence of the universality property for the semantic type $\pounds(T^u_{c.a.})$ together with transitivity of the relation of being a segment of an algebra.

(c)⇔(d) Part ⇒ is followed from Exercise 3.9(d) and [10, Lem. 10.1(e)], while the back implication is obvious.

(d)⇒(f) Given a semantic type $\mathfrak{B} = \bigotimes^{[R]}_{i\in\mathbb{N}} \mathfrak{B}_i$ satisfying properties pointed out in (f). By Exercise 3.9(e), we have $\pounds(T^u_{c.a.}) \equiv_{ASL} \bigotimes^{[EQC^*]}_{i\in\mathbb{N}} \mathfrak{B}'_i$ for an $\mathcal{E}$-rich under ASL computable sequence of $\mathcal{E}$-types. By Lemma 1.5(b), the sequence $\mathfrak{B}'_i$, $i\in\mathbb{N}$, is $\mathcal{E}$-rich under K. By [10, Lem. 9.14], EQC^* is an *inf*-dense theory under the layer $E \subseteq ASL$. By applying Lemma 3.6 and then Lemma 7.3, we obtain

$$\mathfrak{B} = \bigotimes^{[R]}_{i\in\mathbb{N}} \mathfrak{B}_i \equiv_K \bigotimes^{[R]}_{i\in\mathbb{N}} \mathfrak{B}'_i \equiv_K \bigotimes^{[EQC^*]}_{i\in\mathbb{N}} \mathfrak{B}'_i \equiv_{ASL} \pounds(T^u_{c.a.}),$$

that is exactly what is required.

(f)⇒(e) The statement becomes an obvious fact if we prove that there is at least one $\mathcal{E}$-rich under K sequence and an *inf*-dense under E theory. By [10, Lem. 9.14], theory EQC^* is *inf*-dense under ASL. By applying Exercise 3.9(e) together with Lemma 1.5(b) and Lemma 5.4(b), we obtain that the pointed out sequence and an *inf*-dense theory indeed exist.

(e)⇒(d) Is provided by Lemma 7.5.

Theorem 7.5 is proved. □

Theorem 7.5 represents an advanced and extended version of the main statement of the paper [1, Th. 6.1].

REMARK 7.6. Theorem 7.5 concerns structure of computable axiomatizable theories and semantic types. Semantic layer ASL is involved in statements of Theorem 7.5 because all principal methods here are based on singleton (constant) extensions of theories. By applying an available version of the universal construction, we can transform computable axiomatizable theories and semantic types into finitely axiomatizable ones, cf. [10, Th. 6.2]; by that, the controlled layer of model-theoretic properties becomes limited with the layer $\mathcal{MQL}$, cf. (0).

8 Perfection of the concepts of density of theories

We present two statements showing completeness of the concepts of density.

Theorem 8.1. [PERFECTION OF THE CONCEPT OF *f*-DENSITY] *Given a semantic layer $L \subseteq ACL$. For any $\mathcal{F}$-rich under D sequence $\mathfrak{B}_k$ $(k \in \mathbb{N})$ of $\mathcal{F}$-types with $D \supseteq L$, any complete theory P, and any abstract semantic type $\mathfrak{B}$, from $\mathfrak{B} \equiv_N \bigotimes^{[P]}_{k\in\mathbb{N}} \mathfrak{B}_k$ with $N \supseteq L$, we have*:

$$\mathfrak{B} \textit{ is an } \mathcal{F}\textit{-type under} L \;\Leftrightarrow\; (\exists \textit{ an } f\textit{-dense under } L \textit{ theory } P')\big[P' \equiv_L P\big]. \quad (8.1)$$

PROOF. By applying property (1.2) and Lemma 1.5, we can consider just the case $D = N = L$. Implication ⇒ in (6.4) is a corollary of Exercise 5.8, while the back implication is a consequence of Lemma 6.1 together with Theorem 3.5. □

Theorem 8.2. [PERFECTION OF THE CONCEPT OF *inf*-DENSITY] *Given a semantic layer $K \subseteq \mathcal{MQL}$. For any $\mathcal{E}$-rich under E sequence $\mathfrak{B}_k$ $(k \in \mathbb{N})$ of $\mathcal{E}$-types with $E \supseteq L$, any complete theory R, and any abstract semantic type $\mathfrak{B}$, from $\mathfrak{B} \equiv_M \bigotimes^{[P]}_{k\in\mathbb{N}} \mathfrak{B}_k$ with $M \supseteq L$, we have*:

$$\mathfrak{B} \textit{ is an } \mathcal{E}\textit{-type under} L \;\Leftrightarrow\; (\exists \textit{ an } f\textit{-dense under } L \textit{ theory } R')\big[R' \equiv_L R\big]. \quad (8.2)$$

PROOF. By applying property (1.2) and Lemma 1.5, we can consider just the case $E = M = L$. Implication ⇒ in (6.5) is a corollary of Exercise 5.9, while the back implication is a consequence of Lemma 7.1 together with Theorem 3.6. □

Conclusion

A problem of characterization of the Tarski-Lindenbaum algebras of predicate calculi of finite rich signatures was studies in the works [12], [13], [14], and [15]. Some historical background concerning characterization of the Tarski-Lindenbaum algebras of different elementary theories, including predicate calculi of finite rich signatures can be found in the works [16], [12], [14], and [15]. In the work [12], p. 133, it is noted that this problem was initiated by Alfred Tarski in the late 1930s who spent some effort to investigate some aspects of the problem.

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